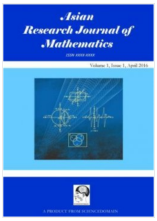

Asian Research Journal of Mathematics

8(1): 1-12, 2018; Article no.ARJOM.38453

ISSN: 2456-477X

The Solution of Fuzzy Integral Equation by Using Sinc Collocation Method

A. Hosseinpour[1*]

[1] *Young Researchers and Elite Club, Kermanshah Branch, Islamic Azad University, Kermanshah, Iran.*

Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/ARJOM/2018/38453

Editor(s):

(1) Radoslaw Jedynak, Department of Computer Science and Mathematics, Kazimierz Pulaski University of Technology and Humanities, Poland.

Reviewers:

(1) Omar Abu Arqub, Al-Balqa Applied University, Jordan.

(2) Rashmi Awad, Devi Ahilya University, India.

Complete Peer review History: http://www.sciencedomain.org/review-history/22680

Received: 27th November 2017

Accepted: 29th December 2017

Published: 10th January 2018

Original Research Article

Abstract

The present paper recommends a new approach to solve a fuzzy integral equation. In this work, our main purpose is finding an approximate solution of this problem by using Sinc-collocation method. We first present some properties of the Sinc-collocation method required for our work, then some basic concepts and necessary materials of fuzzy numbers are expressed in the third part of this paper. In the next section, we apply our method for fuzzy integral equation and show that the approximate solution converges to the exact solution at an exponential rate. Numerical examples given in the last part confirm the accuracy and validity of our new technique. In this example, we obtain the Hausdorff distance between exact solution and approximate solution.

Keywords: Fuzzy integral equations; Sinc-collocation methods; Hilbert space; Conformal map; Whittakers cardinal function.

2010 Mathematics Subject Classification: 65M06, 65N06.

**Corresponding author: E-mail: a_hosseinpour@outlook.com*

1 Introduction

This paper is concerned with a numerical solution of fuzzy Fredholm integral equation. The concept of fuzzy integral equations has been considered by various authors and has been developed rapidly. Ralescu and Adams in [1] presented fuzzy integral of a positive measurable function with respect to a fuzzy measure. They showed that the monotone convergence theorem and Fatous lemma are true in new setting. In 1982, Dubois and Prade published a paper in which they investigated the concept of integration of fuzzy functions. In this paper, they define the integral of such fuzzy mappings over a crisp interval and provide a special analytical representation of fuzzy mapping [2]. In [3], the auto-continuity of a set function and Sugeno's fuzzy measure with some conditions are introduced by Wang. He proved Egoroffs theorem on a fuzzy measure space and some convergence theorems of sequence of fuzzy integrals. Goetschel and Voxman defined differentiation and integration of fuzzy-valued functions by using the usual vector space operations with a spatial metric in [4]. Nanda in [5] generalized the integral introduced by Matloka [6] and also extended Riemann-Stieltjes integral over a closed interval to fuzzy mappings. However, several attempts have been made to apply numerical methods to solve fuzzy integral equations. These techniques have been rapidly developed in recent years. The interested reader can see [7, 8, 9, 10, 11] for some more provided methods to solve fuzzy integral equations. In the present paper, we apply Sinc-collocation method to solve fuzzy Fredholm integral equations. Sinc-collocation method is one of the new and powerful techniques which gives fast and accurate solutions. Sinc-collocation method developed by Frank Stenger, [12, 13, 14], Lund [15] and other authors [16, 17, 18, 19]. In recent years, some authors have been tried to generalized the concept of Sinc methods to fractional calculus. In [20], Fractional integrals, fractional derivatives, fractional integral equations, and fractional differential equations are numerically solved by Sinc methods. For more information about fractional calculus you can see [21, 22, 23, 24, 25, 26, 27]. In this study, we first introduce the Sinc function with its properties and then we state some properties and definitions of fuzzy numbers. However, we apply Sinc collocation method to solve fuzzy Fredholm integral equation. Finally, the efficiency and accuracy of this method is shown by some examples.

2 Definitions and Basic Results

Let $\mathcal{C}$ denote the set of complex numbers. A function f is analytic at $z_0 \in \mathcal{C}$ if its derivative exists at each point in some neighborhood of z_0. If f is analytic in all of $\mathcal{C}$, then f is called entire.

Definition 2.1. The following function defined for all $z \in \mathcal{C}$ is called the Sinc function (See Fig. 1).

$$\mathrm{Sinc}(z) = \begin{cases} \frac{\sin(\pi z)}{\pi z} & z \neq 0, \\ 1 & z = 0. \end{cases} \tag{2.1}$$

Obviously, this is an entire function and arises frequently in the theory of Fourier transforms.
Let f be a real function and h be a positive integer. Set

$$S(k,h) = \mathrm{Sinc}\left(\frac{x-kh}{h}\right), \quad k = 0, \pm 1, \pm 2, \ldots.$$

The series $\sum_{k=-\infty}^{\infty} f(kh)S(k,h)$ is called the cardinal series of function f with respect to the integer h. If this series converges, its sum is denoted by $C(f,h)(x)$ and is called Whittaker cardinal function of f. Note that this function was first studied by E. T. Whittaker [28]. The Cardinal function of f is given by

$$C(f,h)(x) = \sum_{k=-\infty}^{\infty} f(kh)S(k,h). \tag{2.2}$$

Clearly, the cardinal function interpolates f at the points $\{kh\}_{k=-\infty}^{\infty}$. This function appears in various and extensive areas. Stenger in [14] applied cardinal function to derive new approximations for interpolating, integrating and approximating the Fourier series over $(-\infty,\infty)$ and Hilbert transforms over $(-\infty,\infty)$, $(0,\infty)$ and $(-1,1)$. Moreover, this function is used to approximate the derivations of functions over $(-\infty,\infty)$, $(0,\infty)$ and $(-1,1)$ [29].

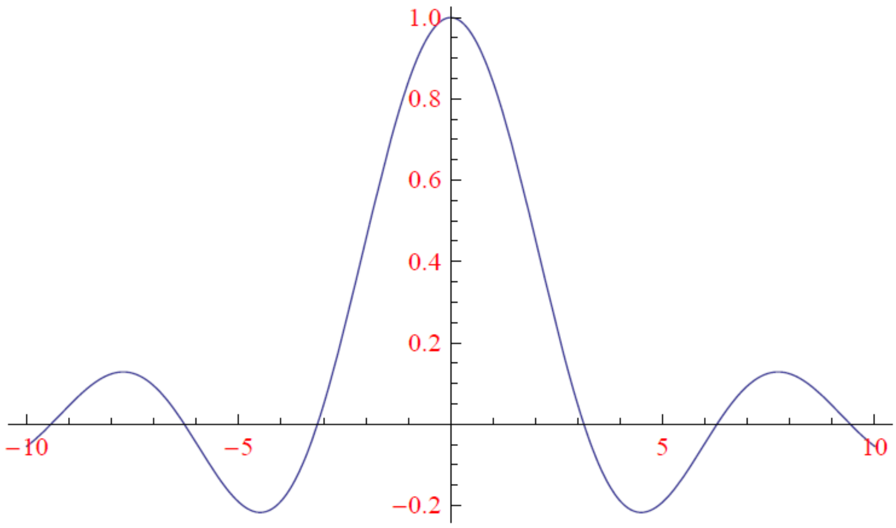

Fig. 1. The graph of a Sinc function

Let h be a positive constant. Let $B(h)$ denote the set of all entire functions f such that

$$|f(z)| \leq K\exp(\frac{\pi|z|}{h}) \quad \textit{and} \quad \left\{\int_a^b |f(x)|^2 dx\right\}^{1/2} < \infty,$$

for some constant $K > 0$. Let us recall an important theorem from [30] showing that when a function is approximated by cardinal series.

Theorem 2.1. *If $f \in B(h)$, then for all $z \in \mathcal{C}$*

$$f(z) = C(f,h)(z) = \sum_{k=-\infty}^{\infty} f(kh)\mathrm{Sinc}(\frac{z-kh}{h}).$$

Definition 2.2. Let f be analytic in a domain D and $z_0 \in D$. Then f is called conformal at z_0 if $f^{'}(z_0) \neq 0$. If $f^{'}(z_0) \neq 0$ for all points in domain D, then f is called a conformal mapping of D.

Remark 2.1. The following statements are satisfied:

1. If function f is conformal, then it preserves angles.
2. If $f : D \to \mathcal{C}$ is conformal and one-to-one, then f^{-1} is conformal where f^{-1} is the inverse of f.
3. If $f : D \to \mathcal{C}$ and $g : \mathcal{C} \to \mathcal{C}$ are conformal maps, then $gof(z) = g(f(z))$ is also a conformal map of z.

The following linear fractional transformation is a specific class of conformal maps of $\mathcal{C}\backslash\{-d/c\}$ onto $\mathcal{C}\cup\{\infty\}$

$$L(z)=\frac{az+b}{cz+d},\qquad ad-bc\neq 0,$$

since $L(z)$ is analytic on $\mathcal{C}\backslash\{-d/c\}$ and $L^{'}(z)\neq 0$, for all $z\in\mathcal{C}\backslash\{-d/c\}$. However, with regard to Remak (2.1) the following map

$$\varphi oL:\mathcal{D}_W\rightarrow\mathcal{D}_S$$
$$\varphi(L(z))=\ln(\frac{az+b}{cz+d})$$

is conformal, where

$$\begin{aligned}\mathcal{D}_W &= \big\{z\in\mathcal{C}\mid \arg(\frac{az+b}{cz+b})<d\big\},\\ \mathcal{D}_S &= \big\{w\in\mathcal{C}\mid w=x+iy,\ |Img(w)|<d\big\}.\end{aligned}$$

Definition 2.3. Two domains D and W are called conformally equivalent if there exists a conformal one to one map φ of D onto W.

In the following, let us introduce some basic concepts of Sinc approximation. Let φ be a conformal one to one map of a simply connected domain $\mathcal{D}$ onto

$$\mathcal{D}_S=\{x+iy:|y|\leq d\},\tag{2.3}$$

where d is a positive constant. Let us denote the boundary of $\mathcal{D}$ by $\partial\mathcal{D}$. Let a, b be two distinct points of $\partial\mathcal{D}$ and assume that $\varphi(a)=-\infty$ and $\varphi(b)=\infty$. Denote the inverse function φ^{-1} by ψ and let Γ be defined by $\Gamma=\big\{z\in\mathcal{C}\mid z=\psi(x)\ ,\ x\in R\big\}=\psi(R)$. For $h>0$, let us define the Sinc points z_k by $z_k=\psi(kh),\ k=0,\pm1,\pm2,\ldots$. Note that the domains $\mathcal{D}$ and $\mathcal{D}_S$ are conformally equivalent. In this research, the approximation of a function is studied by using a finite number of terms of (2.2). In 1949, Goodwin [31] discovered that the accuracy of the approximation of series (2.2) will increase for those functions which are analytic in the strip $\mathcal{D}_S=\{x+iy:|y|\leq d\}$ and converge rapidly to zero as $x\rightarrow\pm\infty$.

Now let us give an important class of functions applied in quadrature rule. Let $B(\mathcal{D}_W)$ denote the family of functions F which are analytic in $\mathcal{D}_W$ and satisfy

$$\int_{\psi(u+L)}|F(z)dz|\rightarrow 0,\qquad u\rightarrow\pm\infty,$$

where $L=\big\{iv\ :\ |v|<d\leq\frac{\pi}{2}\big\}$ and on the boundary of $\mathcal{D}_W$ (denoted by $\partial\mathcal{D}_W$) satisfy

$$\mathcal{N}(F)\equiv\int_{\partial\mathcal{D}_W}|F(z)dz|<\infty.$$

In the what follows, we introduce the Sinc trapezoidal quadrature rule from [15]. This rule which is stated in the next corollary, is very important in our research.

Corollary 2.2. *[32] Let $F\in B(\mathcal{D}_W)$ and φ be a conformal map with constants α and C such that*

$$|\frac{F(z)}{\varphi^{'}(z)}|\leq C\ e^{-\alpha|\varphi(z)|},\quad z\in\Gamma,$$

then by selecting $h=\sqrt{\pi d/\alpha N}$ the Sinc trapezoidal quadrature rule is expressed as bellow

$$\int_\Gamma F(z)S(k,h)o\phi(z)dz=h\frac{F(z_k)}{\phi^{'}(z_k)}+O(\exp(-(2\pi d\alpha N)^{\frac{1}{2}})).\tag{2.4}$$

3 Fuzzy Set Properties

A nonempty subset B of R is called fuzzy convex if and only if $u(\lambda x+(1-\lambda)y) \geq min\{u(x), u(y)\}$, for every $x, y \in B$ and $\lambda \in [0,1]$. Let $H(R^n)$ denote the space of non-empty compact convex subsets of R^n. Recall that for any $A \subseteq H(R^n)$ we define

$$d(x, A) = \inf_{a \in A} d(x, a)$$

to be the distance of any $x \in R^n$ from A, and for any positive number ε, we define the ε-neighborhood of A as the set

$$N(A, \varepsilon) = \big\{x \in R^n |\ d(x, A) < \varepsilon\big\} = \bigcup_{a \in A} \big\{x \in R^n |\ d(x, a) < \varepsilon\big\}.$$

Notice that the infimum in the definition of $d(x, A)$ is actually achieved, that is, there is a point $a \in A$ such that $d(x, A) = d(x, a)$, because A is compact. For A and $B \in H(R^n)$, the Hausdorff separation $\rho(A, B)$ is defined as bellow

$$\rho(A, B) = \inf \big\{\varepsilon > 0 |\ A \subseteq N(B, \varepsilon)\big\}.$$

The Hausdorff metric on $H(R^n)$ is introduced as bellow

$$\begin{aligned} d_H(A, B) &= \max \big\{\rho(A, B), \rho(B, A)\big\} \\ &= \inf \big\{\varepsilon |\ A \subseteq N(B, \varepsilon) \ and \ B \subseteq N(A, \varepsilon)\big\}. \end{aligned}$$

It is obvious that $(H(R^n), d_H)$ is a complete and separable metric space [33].

Definition 3.1. A fuzzy number is a function such as $u : R^n \to [0,1]$ satisfying the following properties:

1. u is normal, i.e. $\exists\ x_0 \in R^n$ such that $u(x_0) = 1$,
2. u is fuzzy convex,
3. u is upper semicontinuous,
4. $[u]^0 = \overline{\{x \in R^n : u(x) > 0\}}$ is compact.

The set of all fuzzy numbers is denoted by E^n. If u is a fuzzy number in R^n, we define $[u]^\alpha = \overline{\{x \in R^n |\ u(x) > \alpha\}}$ the α-level of u with $0 < \alpha \leq 1$. For $\alpha = 0$ the support of u is defined as $[u]^0 = supp(u) = \overline{\{x \in R^n \mid u(x) > 0\}}$. Clearly, for any $\alpha \in [0,1]$, $[u]^\alpha$ is a bounded closed interval. For all $u, v \in E^n$ and for any real number λ we obtain

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha \quad , \quad [\lambda u]^\alpha = \lambda [u]^\alpha, \qquad \forall \alpha \in [0,1].$$

In the what follows, we present another definition for a fuzzy number which will be used in the rest of this paper.

Definition 3.2. An arbitrary fuzzy number is showed by an ordered pair of functions $(\underline{u}(r), \overline{u}(r)),\ 0 \leq r \leq 1$, which satisfies the following requirements:

1. $\underline{u}(r)$ is a bounded left semicontinuous non-decreasing function over $[0,1]$,
2. $\overline{u}(r)$ is a bounded left semicontinuous non-increasing function over $[0,1]$,
3. $\underline{u}(r) \leq \overline{u}(r),\ 0 \leq r \leq 1$.

In particular, if $\underline{u}$, $\overline{u}$ are linear functions we have a triangular fuzzy number. A crisp number u is simply represented by $\underline{u}(r) = \overline{u}(r) = u,\ 0 \leq r \leq 1$. For arbitrary fuzzy numbers $u = (\underline{u}(r), \overline{u}(r)),\ v = (\underline{v}(r), \overline{v}(r))$ we have the following algebraic operations:

1. $ku = \begin{cases} (k\underline{u}, k\overline{u}) & k \geq 0, \\ (k\overline{u}, k\underline{u}) & k < 0, \end{cases}$
2. $u + v = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)),$
3. $u - v = (\underline{u}(r) - \overline{v}(r), \overline{u}(r) - \underline{v}(r)).$

We now introduce the concepts of fuzzy Differentiability and integrability. Two following definitions are due to Puri and Ralesco in [34] and R. Goetschel and W. Voxman in [4].

Definition 3.3. A fuzzy function $F : (a, b) \to R^n$ is called differentiable at $t_0 \in (a, b)$ if there exists $F^{'}(x_0) \in R^n$, such that the limits

$$\lim_{h \to 0^+} \frac{F(t_0 + h) - F(t_0)}{h} \quad and \quad \lim_{h \to 0^+} \frac{F(t_0) - F(t_0 - h)}{h}$$

both exist and are equal to $F^{'}(x_0)$.

Definition 3.4. Let $f : [a, b] \to E^1$ be a fuzzy function. For each partition $P = \{t_0, t_1, \ldots, t_n\}$ of $[a, b]$ with $\Delta = \max_{i=1,\ldots,n} |t_i - t_{i-1}|$ and for arbitrary $\xi_i \in [t_{i-1}, t_i]$ $(i = 1, \ldots, n)$, set

$$R_P = \sum_{i=1}^{N} f(\xi_i)(t_i - t_{i-1}).$$

The definite integral of f over $[a, b]$ is defined as bellow

$$\int_a^b f(t)dt = \lim_{\Delta \to 0} R_P$$

If the fuzzy function f is continuous in the metric D, then this integral exists. Furthermore,

$$\overline{\int_a^b f(t, r)\ dt} = \int_a^b \overline{f}(t, r)\ dt,$$

$$\underline{\int_a^b f(t, r)\ dt} = \int_a^b \underline{f}(t, r)\ dt.$$

Note that the definition of fuzzy Riemann integral is stated here. The interested reader can refer to [1] for definitions and some properties of fuzzy Lebesgue integral, you can also see the fuzzy integral of a positive, measurable function with respect to a fuzzy measure, the monotone convergence theorem and Fatous lemma in the space of fuzzy number.

4 Fuzzy Fredholm Integral Equations

In this section, we want to solve the following fuzzy integral equations

$$\widetilde{u}(x) = \widetilde{g}(x) + \lambda \int_a^b k(x, t)\widetilde{u}(t)dt, \tag{4.1}$$

where $\lambda > 0$ and $k(x, t)$ is an arbitrary kernel function over the square $a \leq x, t \leq b$ and $\widetilde{g}(x)$ is a fuzzy function of x. Let $\widetilde{u}(x)$ is an unknown fuzzy function in the space $C[a, b]$ of continuous functions on the closed interval $[a, b]$. We now introduce parametric form of Equation (4.1) as follows:

$$\underline{u}(x, r) = \underline{g}(x, r) + \begin{cases} \int_a^b k(x, t)\underline{u}(t, r)dt & k(x, t) > 0 \\ \int_a^b k(x, t)\overline{u}(t, r)dt & k(x, t) < 0 \end{cases}$$

$$\overline{u}(x,r)=\overline{g}(x,r)+\begin{cases}\int_a^b k(x,t)\overline{u}(t,r)dt & k(x,t)>0\\ \int_a^b k(x,t)\underline{u}(t,r)dt & k(x,t)<0\end{cases}$$

for each $0 \le r \le 1$ and $a \le x,t \le b$. For solving these integral equations we apply Sinc-collocation method. Let φ be a conformal map on the interval (a,b). Consider the following basic functions on $\mathcal{C}$

$$\begin{aligned}\chi_k(x) &= S(k,h)o\varphi(x)\\ &= \operatorname{sinc}\left(\frac{\varphi(x)-kh}{h}\right), \quad k \in Z.\end{aligned}$$

Obviously, $\{\chi_k\}_{k=1}^{\infty}$ is a dense linearly independent set in $\mathcal{C}$, set $H_N = span\{\chi_1,\dots,\chi_N\}$. However, each function $\widetilde{u} \in H_N$ is approximated by

$$\widetilde{u}_m(x)=\sum_{k=-M}^{N}\widetilde{u}_k\chi_k(x), \qquad m=M+N+1,$$

where N is a positive integer number and the coefficients $\widetilde{u}_k$ should be determined. Therefore, in Equation (4.1), all functions are approximated as bellow

$$\begin{aligned}\widetilde{u}(x) &= \sum_{j=-N}^{N}\widetilde{u}_j S(j,h)o\varphi(x),\\ k(x,t) &= \sum_{j=-N}^{N}\sum_{i=-N}^{N}k_{j,i}S(j,h)o\varphi(x)S(i,h)o\varphi(t),\\ \widetilde{g}(x) &= \sum_{j=-N}^{N}\widetilde{g}_j S(j,h)o\varphi(x).\end{aligned}$$

By substituting these functions in Equation (4.1), we obtain

$$\begin{aligned}\sum_{j=-N}^{N}\widetilde{u}_j S(j,h)o\varphi(x) &= \sum_{j=-N}^{N}\widetilde{g}_j S(j,h)o\varphi(x)\\ &+ \int_0^1\sum_{j=-N}^{N}\sum_{i=-N}^{N}k_{j,i}P_iS(j,h)o\varphi(x)S(i,h)o\varphi(t)dt.\end{aligned}$$

Using Equation (2.4), we have

$$\sum_{j=-N}^{N}\left[\widetilde{u}_j-\sum_{i=-N}^{N}h\frac{k_{ji}}{\varphi_i'}\right]S(j,h)o\varphi(x)=\sum_{j=-N}^{N}\widetilde{g}_j S(j,h)o\varphi(x).$$

These equations can be simplified in the matrix form as bellow:

$$\widetilde{U}-K=\widetilde{G}, \tag{4.2}$$

where

$$\widetilde{U}=\begin{pmatrix}\widetilde{u}_N\\ \vdots\\ \widetilde{u}_{-N}\end{pmatrix}, \quad \widetilde{G}=\begin{pmatrix}\widetilde{g}_N\\ \vdots\\ \widetilde{g}_{-N}\end{pmatrix}$$

and $K=[\frac{k_{ji}}{\varphi_i'}]$ is a square matrix of order $(2N+1)\times(2N+1)$. The system (4.2) provides a set of $2N+1$ equations which can be calculated for the unknown coefficients $(\widetilde{u}_{-N},\dots,\widetilde{u}_N)$.

5 Numerical Results

At this stage, we need to select a conformal map and use it to find the Sinc grid points. In this research, we consider the conformal map

$$\varphi(z) = \ln\left(\frac{z}{1-z}\right) \tag{5.1}$$

which maps the region

$$\mathcal{D}_W = \left\{ z \in \mathcal{C} \mid \arg\left(\frac{z}{1-z}\right) < d \right\}$$

onto the infinite strip $\mathcal{D}_S$ defined in (2.3). The function

$$z = \varphi^{-1}(w) = \frac{e^w}{1+e^w}$$

is an inverse mapping of $w = \varphi(z)$. The range of $\psi = \varphi^{-1}$ on the real line is defined as

$$\Gamma = \left\{ \psi(u) = \varphi^{-1}(u) \in \mathcal{D}_W \mid u \in R \right\} = (0, 1).$$

The Sinc grid points $z_k \in (0, 1)$ in $\mathcal{D}_W$ will be denoted by x_k because they are real. As mentioned in Section 2, these nodes are obtained by

$$x_k = \varphi^{-1}(kh) = \frac{e^{kh}}{1+e^{kh}}.$$

In the following, let us present some fuzzy integral equations to show the efficiency of the method proposed in the previous sections. In the next examples, put $\alpha = 1/2$ and $d = \pi/2$ which leads to $h = \pi/\sqrt{N}$.

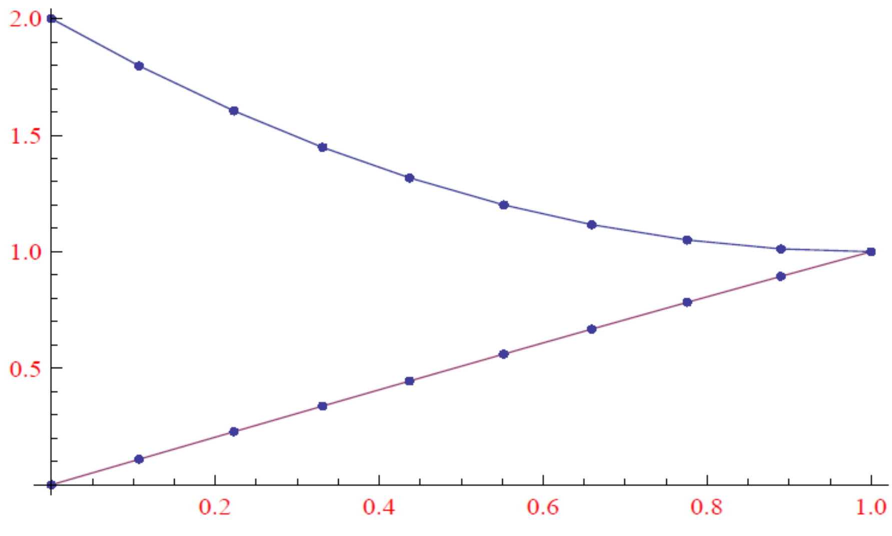

Fig. 2. The exact and approximate solutions

Example 5.1. *Consider the following integral equation*

$$\widetilde{u}(x, r) = \widetilde{g}(x, r) + \int_0^{\pi/2} (x + t)\widetilde{u}(t, r)dt,$$

where

$$\underline{g}(x,r) = (r^2 - 2r + 2)(\sin x - x - 1),$$
$$\overline{g}(x,r) = (\sin \frac{\pi r}{6} + \frac{r}{2})(\sin x - x - 1).$$

The exact solution in this case is given by

$$\underline{u}(x,r) = (r^2 - 2r + 2)\sin x$$
$$\overline{u}(x,r) = (\sin \frac{\pi r}{6} + \frac{r}{2})\sin x$$

see Fig. 2.

Example 5.2. *Consider the following integral equation*

$$\widetilde{u}(x,r) = \widetilde{g}(x,r) + \int_0^1 e^{x^2+t}\, \widetilde{u}(t,r)dt,$$

where

$$\underline{g}(x,r) = (-r^2 + 2)(x + 1 - e^{x^2+1}),$$
$$\overline{g}(x,r) = \frac{1}{2}(\sin^2 \frac{\pi r}{2} + 1)[1 - 2x^2 - (3 - e)e^{x^2}].$$

The exact solution of the above equation is

$$\underline{u}(x,r) = (-r^2 + 2)(x + 1),$$
$$\overline{u}(x,r) = \frac{1}{2}(\sin^2 \frac{\pi r}{2} + 1)(1 - 2x^2).$$

see Fig. 3.

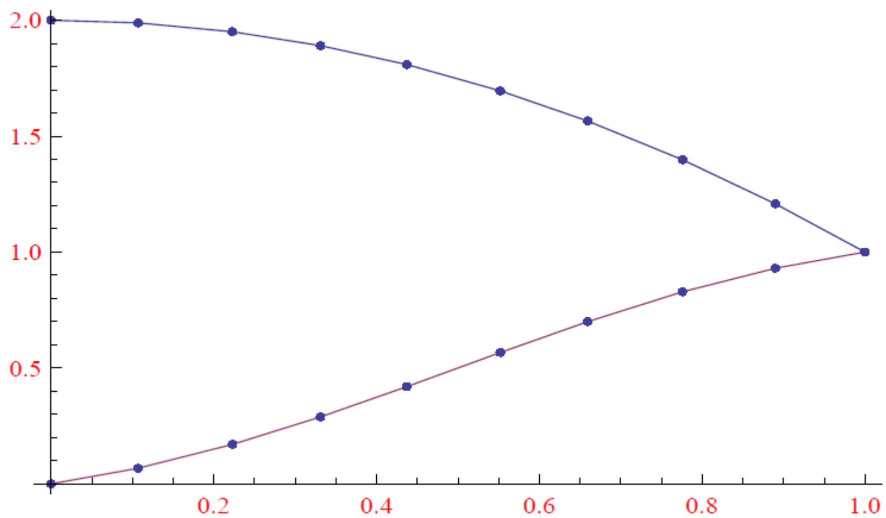

Fig. 3. The exact and approximate solutions

6 Conclusions

The purpose of the current study is to apply the Sinc-collocation method for solving fuzzy Fredholm integral equation. This method is based on Sinc basis functions. The main advantage of this method is that the order of convergence is at an exponential rate. However, properties of Sinc-collocation method have been employed to transform fuzzy integral equation to an explicit form of algebraic equations which is easy to accomplish. The numerical examples in the last section verify the efficiency of our procedure.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Ralescu D, Adams G. The fuzzy integrals. J Math Anal Appl. 1980;75:562-570.

[2] Dubois D, Prade H. Towards fuzzy differential calculus part 1: Integration of fuzzy mappings. Fuzzy Sets and Systems. 1982;8:1-17.

[3] Wang Z. The autocontinuity of set-function and the fuzzy integral. J Math Anal Appl. 1984;99:195-218.

[4] Goetschel R, Voxman W, Elementary fuzzy calculus, Fuzzy Sets Syst. 1986;18:31-43.

[5] Nanda S. On integration of fuzzy mappings. Fuzzy Sets Syst. 1989;32(32):95-101.

[6] Matloka M. On fuzzy integrals, Proc. 2nd Polish Symp. on Interval and Fuzzy Mathematics, Politechnika Poznansk. 1987;167-170.

[7] Friedman M, Ming M, Kandel A. Numerical solutions of fuzzy differential and integral equations. Fuzzy Sets and Systems. 1999;106:35-48.

[8] Maleknejad K, Mahmodi Y. Numerical solution of linear Fredholm integral equation by using hybrid Taylor and Blockpulse functions. Appl Math Comput. 2004;149:799-806.

[9] Wu H. The fuzzy Riemann integral and its numerical integration. Fuzzy Sets Syst. 2000;110:1-25.

[10] Kanwal RP, Liu KC. A Taylor expansion approach for solving integral equations. International Journal of Mathematical Education in Science and Technology. 1989;20:411-414.

[11] Sorkun HH, Yalcinbas S. Approximate solutions of linear Volterra integral equation systems with variable coefficients. Applied Mathematical Modelling. 2010;34(11):3451-3464.

[12] Stenger F. Numerical methods based on Sinc and analytic functions. New York: Springer-Verlag; 1993.

[13] Stenger F. Summary of sinc numerical methods. J. Comput. Appl. Math. 2000;121:379-420.

[14] Stenger F. Approximations via Whittaker's cardinal function. Journal of Approximation Theory. 1979;17:222-240.

[15] Lund J, Bowers K. Sinc methods for quadrature and differential equations. SIAM Philadelphia; 1992.

[16] Sugihara M, Matsuo T. Recent developments of the Sinc numerical methods. J. Comput. Appl. Math. 2004;164(165):673-689.

[17] Winter DF, Bowers K, Lund J. Wind-driven currents in a sea with a variable Eddy viscosity calculated via a Sinc-Galerkin technique. Internt. J. Numer. Methods Fluids. 2000;33:1041-1073.

[18] Stenger F. Numerical methods based on Whitaker cardinal or Sinc functions. SIAM Review. 1981;23(2):165-224.

[19] Muhammad M, Mori M. Double exponential formulas for numerical indefinite integration. J. Comput. Appl. Math. 2003;161:431-448.

[20] Baumann G, Stenger F. Fractional calculus and Sinc methods. An International Journal for Theory and Application. 2011;14(4):568-622.

[21] Abu Arqub O, El-Ajou A, Momani S. Constructing and predicting solitary pattern solutions for nonlinear time-fractional dispersive partial differential equations. Journal of Computational Physics. 2015;293:385-399.

[22] El-Ajou A, Abu Arqub O, Momani S, Baleanu D, Alsaedi A. A novel expansion iterative method for solving linear partial differential equations of fractional order. Applied Mathematics and Computation. 2015;257:119-133.

[23] El-Ajou A, Abu Arqub O, Momani S. Approximate analytical solution of the nonlinear fractional KdV-Burgers equation: A new iterative algorithm. Journal of Computational Physics. 2015;293:81-95.

[24] Abu Arqub O. Numerical solutions for the Robin time-fractional partial differential equations of heat and fluid flows based on the reproducing kernel algorithm. International Journal of Numerical Methods for Heat and Fluid Flow; 2017.
DOI: 10.1108/HFF-07-2016-0278.

[25] Abu Arqub O. Fitted reproducing kernel Hilbert space method for the solutions of some certain classes of time-fractional partial differential equations subject to initial and Neumann boundary conditions. Computers Mathematics with Applications. 2017;73:1243-1261.

[26] Abu Arqub O, Al-Smadi M, Momani S, Hayat T. Numerical solutions of fuzzy differential equations using reproducing kernel Hilbert space method. Soft Computing. 2016;20:3283-3302.

[27] Abu Arqub O, Al-Smadi M, MomaniS, Hayat T. Application of reproducing kernel algorithm for solving second-order, two-point fuzzy boundary value problems. Soft Computing. 2016;1-16.
DOI: 10.1007/s00500-016-2262-3.

[28] Whittaker ET. On the functions which are represented by the expansions of the interpolation theory. Proc. Roy. Soc. Edinburgh. 2014;35(1915):181-194.

[29] Lundin L, Stenger F. Cardinal-type approximations of a function and its derivatives. SIAM J. Math. Anal. 1979;10(1):139160.

[30] McNamee J, Stenger F, Whitney EL. Whittaker's cardinal function in retrospect. Mathematics of Computation. 1971;25(113):141-154.

[31] Goodwin ET. The evaluation of integrals of the form $\int_{-\infty}^{\infty} f(x)e^{-x^2}dx$ Proc. Cambridge Philos. Sot. 1949;45:241-245.

[32] Rashidinia J, Zarebnia M. New approach for numerical solution of Hammerstein integral equations. Applied Mathematics and Computation. 2007;185:147154.

[33] Puri M, Ralescu D. Fuzzy random variables. J. Math. Anal. Appl. 1986;409-422.

[34] Puri M, Ralescu D. Differential and fuzzy functions. J Math Anal Appl. 1938;91:552-8.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/22680