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Numerical Study of Viscous Dissipation and Temperature-Dependent Viscosity on MHD Free Convection Flow over a Sphere with Heat Conduction

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Authors' contributions

This work was carried out in collaboration between all authors. Author MMA designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors RI, MMA, RI, MMI and MHR managed the analyses of the study. Authors RI and MHR managed the literature searches. All authors read and approved the final manuscript.

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Abstract

The effect of viscous dissipation and temperature dependent viscosity on MHD free convection flow over a sphere with heat conduction has been analyzed in the article. The governing equations are transformed into dimensionless non-similar equations by using a set of useful transformations and solved numerically by finite difference method along with Newton's linearization approximation. Attention has been focused on the evaluation of shear stress in terms of local skin friction and local Nusselt number, the velocity and the temperature profiles over the whole boundary layer are shown graphically by using Tecplot-10 and tabular form for different values of the Prandtl's number P_r, magnetic parameter M, the temperature dependent viscosity ε, viscous dissipation parameter N and the Joule heating parameter J.

Keywords: Free convection; Joule-heating; Magneto hydrodynamics; Variation viscosity and viscous dissipation.

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NOMENCLATURES

a	*Radius of the sphere*
C_p	*specific heat*
Cf_x	*Skin friction coefficient*
f	*dimensionless stream function*
g	*acceleration due to gravity*
G_r	*Grashof number*
H_0	*Magnetic field strength*
J	*Joule heating parameter*
M	*Magnetic parameter*
N	*viscous dissipation parameter*
Nu_x	*Nusselt number*
Pr	*Prandtl's number*
r	*Radius of the radical distance*
T	*temperature*
T_w	*temperature at the wall*
T_∞	*Temperature at ambient fluid*
U, V	*velocity components*
u, v	*dimensionless velocity components*
u_0	*characteristic velocity*
X, Y	*Cartesian coordinates*
x, y	*dimensionless Cartesian coordinates*
X/a	*angle of the sphere*

GREEK SYMBOLS

β	*co-efficient of thermal expansion*
ε	*Temperature dependent viscosity parameter*
K	*thermal conductivity*
μ	*dynamic viscosity of the fluid*
μ_∞	*the ambient dynamic viscosity*
ν	*kinematic viscosity of the fluid*
θ	*dimensionless temperature*
ρ	*density of the fluid*
σ	*electrical conductivity*
ψ	*stream function*

1 Introduction

Free convection flow is often encountered in cooling of nuclear reactors or in the study of the structure of stars and planets. Along with the free convection flow the phenomenon of the boundary layer flow of an electrically conducting fluid over a sphere in the presence of a Joule-heating term and magnetic field are also very common because of their applications in nuclear engineering in connection with the cooling of reactors. Numerical study of viscous dissipation and temperature dependent viscosity on magnetohydrodynamic (MHD) free convection flow with heat conduction have been taken on various geometrical shapes such as a vertical flat plate, cylinder, sphere *etc.* studied by many investigators and it has been a very popular research topic for many years. Alam et al. [1] investigated the viscous dissipation effects on MHD natural convection flow over a sphere in the presence of heat generation. The effect of viscous dissipation on natural convection flow alone a sphere with heat generation is considered by Akter, S. et al. [2]. Miraj, M. et al. [3] discussed the conjugate effects of radiation and viscous dissipation on natural convection flow over a sphere with pressure work. Molla, M. M. et al. [4] carried out the effects of temperature-dependent thermal conductivity on MHD natural convection flow from an isothermal sphere. The effects of temperature-dependent thermal conductivity on MHD free convection flow along a vertical flat plate with heat generation and Joule heating examined by Islam, S. et al. [5]. Nasrin, R. et al. [6] performed the combined effects of viscous dissipation and temperature dependent thermal conductivity on magnetohydrodynamic (MHD) free convection flow with conduction and Joule heating along a vertical flat plate. Gitima [7] presented the analysis of the effect of variable viscosity and thermal conductivity in micropolar fluid for a porous channel in presence of magnetic field. Cebeci, T., & Brashaw, P.[8] established the physical and computational aspects of convective heat transfer. Keller H.B. [9] developed the numerical methods in boundary layer theory. Nasrin, R. et al. [10] made the MHD free convection flow along a vertical flat plate with thermal conductivity and viscosity depending on temperature. Nabil Eldabe T.M., et al. [11] explored the effects of temperature-dependent viscosity and viscous dissipation on MHD convection flow from an isothermal horizontal circular cylinder in the presence of stress work and heat generation. Safiqul Islam K. M., *et al.* [12] established the effects of temperature-dependent thermal conductivity on natural convection flow along a vertical flat plate with heat generation. Molla, M. M., *et al.* [13] studied the effect of temperature dependent viscosity on MHD natural convection flow from an isothermal sphere. Alim, M. A. et al. [14] showed the heat generation effects on MHD natural convection flow along a vertical wavy surface with variable thermal conductivity. Md. Raihanul Haque et al. [15] observed the effects of viscous dissipation on natural convection flow over a sphere with temperature-dependent thermal conductivity. Charruaudeau, J. [16] examined the influence de gradients de properties physiques en convection force application au cas du tube. Nazar et al. [17] established the free convection boundary layer on an isothermal sphere in a micro polar fluid. Molla, M.M. et al. [18] performed the natural convection flow along a vertical wavy surface with uniform surface temperature in presence of heat generation / absorption. Md.,M.Alam et al. [19], analyzed the free convection from a vertical permeable circular cone with pressure work and non-uniform surface temperature. Pinakee Dey et al. [20] studied the effects of an asymptotic method for over-damped forced nonlinear vibration systems with slowly varying coefficients.

In the present work, we have investigated the viscous dissipation and temperature dependent viscosity with MHD and Joule heating effect on the skin friction and the local heat transfer coefficient in the entire region from upstream to downstream of a viscous incompressible and electrically conducting fluid over a sphere. The transformed non-similar boundary layer equations governing the flow together with the boundary conditions based on conduction and convection were solved numerically using the implicit finite difference method with Keller box [9] scheme by Cebeci and Bradshaw [8] along with Newton's linearization approximation method. We have studied the effect of the Prandtl's number *Pr,* the viscous dissipation parameter *N*, the Joule heating parameter *J*, Magnetic parameter *M* and variation viscosity parameter ε on the velocity and temperature fields as well as on the skin friction and local heat transfer coefficient. All the investigations for the fluid with low Prandtl's number appropriate for the liquid metals are carried out.

2 Formulation of the Problem

We consider a steady two-dimensional natural convection boundary layer flow of an electrically conducting and viscous incompressible fluid over a sphere, H_0 is the magnetic field strength and σ is the electrical conductivity. The surface temperature of the sphere is assumed as T_w and T_∞ being the ambient temperature of the fluid. When $T_w > T_\infty$ an upward flow is established along the surface due to free convection and the flow is downward for $T_w < T_\infty$.

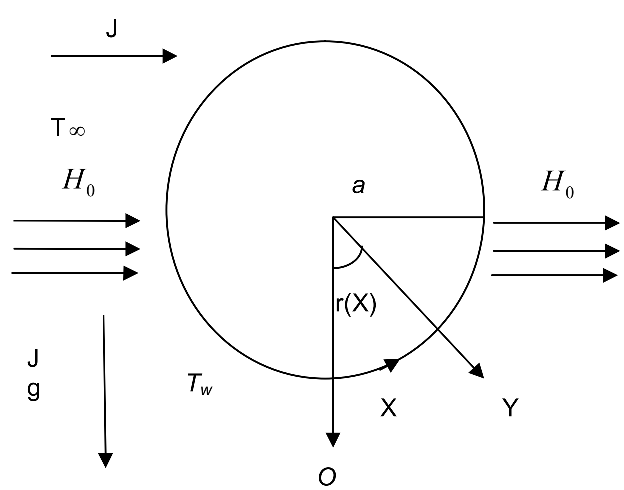


Fig. 1. Physical model and coordinate system

The mathematical model for the assumed physical problem is prescribed by the following conservation equation of mass, momentum and energy. Under these considerations the governing equations are

$$\frac{\partial}{\partial X}(rU)+\frac{\partial}{\partial Y}(rV)=0 \tag{1}$$

$$U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=\frac{1}{\rho}\frac{\partial}{\partial Y}\left(\mu\frac{\partial U}{\partial Y}\right)+g\beta\left(T\text{ - }T_\infty\right)\sin\left(\frac{X}{a}\right)-\frac{\sigma H_0^2 U}{\rho} \tag{2}$$

$$U\frac{\partial T}{\partial X}+V\frac{\partial T}{\partial Y}=\frac{K}{\rho C_p}\frac{\partial^2 T}{\partial Y^2}+\frac{\mu}{\rho C_p}\left(\frac{\partial U}{\partial Y}\right)^2+\frac{\sigma H_0^2}{\rho C_p}U^2 \tag{3}$$

The boundary conditions for the governing equations are

$$\left.\begin{aligned}&U=V=0,\quad T=T_w\quad on\quad Y=0\\&U\to 0, T\to T_\infty\quad at\quad Y\to\infty\end{aligned}\right\} \tag{4}$$

$$r=a\ \sin\left(\frac{X}{a}\right),\ where\ r=r(X) \tag{5}$$

Where a is the radius of the sphere and r is the radial distance from the symmetrical axis to the surface of the sphere. Here we will consider $\mu=\frac{\mu_\infty}{1+\alpha(T-T_\infty)}$ is the dependent viscosity where $\alpha=\frac{1}{\mu_f}\left(\frac{\delta\mu}{\delta T}\right)_f$.

3 Transform of the Governing Equations

The above equations are made dimensionless by using the following substitutions:

$$\begin{aligned}&x=\frac{\mathrm{X}}{a}\ ,\mathrm{y}=\mathrm{Gr}^{1/4}\ \frac{\mathrm{Y}}{a},\mathrm{u}=\frac{U}{u_0}=\frac{\mathrm{a}}{\nu}\mathrm{Gr}^{-1/2}\ \mathrm{U}\ ,\\&v\ =\frac{\mathrm{a}}{\nu}\mathrm{Gr}^{-1/4}V,\theta\ =\frac{\mathrm{T\text{-}T_\infty}}{\mathrm{T_w\text{-}T_\infty}}\end{aligned} \tag{6}$$

Where, $u_0=\frac{\upsilon}{a}Gr^{1/2}$ is the characteristic velocity of the fluids. Here, we will consider

$$\beta=-\frac{1}{\rho}\left(\frac{\delta\rho}{\delta T_f}\right)_p,\ Gr=\frac{g\beta a^3}{\upsilon^2}\left(T_w-T_\infty\right),$$

$$\varepsilon=\frac{1}{\mu_f}\left(\frac{\delta\mu}{\delta T}\right)_f\left(T-T_\infty\right),\ \frac{\mu}{\mu_\infty}=\frac{1}{1+\varepsilon\theta}$$

Using the above transformations into equations (1) to (3), we have

$$\therefore \frac{\delta}{\delta x}(ru)+\frac{\delta}{\delta y}(rv)=0 \tag{7}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{-\varepsilon}{(1+\varepsilon\theta)^2}\frac{\partial u}{\partial y}\frac{\partial \theta}{\partial y}+\frac{1}{1+\varepsilon\theta}\frac{\partial^2 u}{\partial y^2}+\theta\sin x-Mu \tag{8}$$

$$u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=\frac{1}{Pr}\frac{\partial^2 \theta}{\partial y^2}+N\left(\frac{\partial u}{\partial y}\right)^2+J\left(u^2\right) \tag{9}$$

The boundary conditions associated with (9) to (10) become

$$\left.\begin{array}{l} u=v=0,\quad \theta=1 \quad at\ x=0,\ for\ any\ y \\ u=v=0,\quad \theta=1 \quad at\ y=0, x>0 \\ u\to 0,\quad \theta\to 0\ \ as\ y\to\infty,\ x>0 \end{array}\right\} \tag{10}$$

Here, Where, $u_0=\frac{\upsilon}{a}Gr^{1/2}$ is the characteristic velocity of the fluids, $M=\frac{\sigma H_0{}^2 a^2}{\mu_\infty G_r{}^{1/2}}$ is magnetic parameter $Gr=g\beta(T_w-T_\infty)a^3/v^2$ is the Grashof number and θ is the non-dimensional temperature function, $\Pr=\frac{\mu C_p}{k_\infty}$ is the Prandtl's number, and $\varepsilon=\frac{1}{\mu_f}\left(\frac{\delta\mu}{\delta T}\right)_f\left(T-T_\infty\right)$ is the variation viscosity parameter. Also $N=\frac{Gr}{a^2C_p\left(T_w-T_\infty\right)}$ is the viscous dissipation parameter, $J=\frac{\sigma H_o^2 ag\beta\left(T_w-T_\infty\right)}{\rho C_p}$ is the Joule heating parameter. To solve equations (9) and (10) subject to the boundary conditions (11), we assume the following variables $u=\frac{1}{r}\frac{\partial\psi}{\partial y}$ and $v=-\frac{1}{\mathrm{r}}\frac{\partial\psi}{\partial x}$ where $\psi=x\,r(x)\,f(x,y)$ and $\psi(x,y)$ is a non-dimensional stream function which is related to the velocity components in the usual way as $u=\frac{1}{r}\frac{\partial\psi}{\partial y}$ and $v=-\frac{1}{\mathrm{r}}\frac{\partial\psi}{\partial x}$

The momentum and energy equations (9) to (10) reduce to

$$\begin{aligned}&\frac{1}{1+\varepsilon\theta}\frac{\delta^3 f}{\delta y^3}+\left(1+\frac{x}{\sin x}\cos x\right)f\frac{\delta^2 f}{\delta y^2}-\left(\frac{\delta f}{\delta y}\right)^2-\frac{\varepsilon}{(1+\varepsilon\theta)^2}\frac{\delta\theta}{\delta y}\frac{\delta^2 f}{\delta y^2}\\&+\frac{\theta\sin x}{x}-M\frac{\partial f}{\partial y}=x\left(\frac{\delta f}{\delta y}\frac{\delta^2 f}{\delta y\delta x}-\frac{\delta f}{\delta x}\frac{\delta^2 f}{\delta y^2}\right)\end{aligned} \tag{11}$$

$$\begin{aligned}&\frac{1}{\Pr}\frac{\partial^2\theta}{\partial y^2}+\left(1+\frac{x}{\sin x}\cos x\right)f\frac{\partial\theta}{\partial y}+Nx^2\left(\frac{\partial^2 f}{\partial y^2}\right)^2\\&J\left(\frac{\partial f}{\partial y}\right)^2=x\left(\frac{\partial f}{\partial y}\frac{\partial\theta}{\partial x}-\frac{\partial f}{\partial x}\frac{\partial\theta}{\partial y}\right)\end{aligned} \tag{12}$$

The corresponding boundary conditions are

$$\left.\begin{array}{l} f = \dfrac{\partial f}{\partial y} = 0, \quad \theta = 1 \; at \; y = 0 \quad for\,any \; y \\ f = \dfrac{\partial f}{\partial y} = 0, \quad \theta = 1 \; at \; y = 0, x > 0 \\ \dfrac{\partial f}{\partial y} \to 0, \theta \to 0 \; as \; y \to \infty, x > 0 \end{array}\right\} \tag{13}$$

In practical application, the physical quantities of principal interest are the heat transfer and the skin- friction coefficient, which can be written in non- dimensional form as

$$Nu_x = \frac{aGr^{-1/4}}{k\left(T_w - T_\infty\right)} q_w \quad \text{and} \quad Cf_x = \frac{Gr^{-3/4} a^2}{\mu\nu} \tau_w \tag{14}$$

Where $q_w = -k\left(\frac{\partial T}{\partial Y}\right)_{Y=0}$ and $\tau_w = \mu\left(\frac{\partial U}{\partial Y}\right)_{Y=0}$, k being the thermal conductivity of the fluid. Using the new variables (6), we have the simplified form of the heat transfer and the skin- friction coefficient as

$$\theta'(x,0) = Nu_x = -\left(\frac{\partial \theta}{\partial y}\right)_{y=0} \quad \text{and} \quad f''(x,0) = Cf_x = x\left(\frac{\partial^2 f}{\partial y^2}\right)_{y=0} \tag{15}$$

4 Method of Solution

This paper deals with the natural convection flow on variation viscosity and viscous dissipation of viscous incompressible fluid over a heated sphere with Joule heating and magneto hydrodynamics being investigated using the very efficient implicit finite difference method known as the Keller box scheme developed by Keller [9], which has been well documented by Cebeci and Bradshaw [8].

To apply the aforementioned method, equations (11) and (12) with their boundary conditions (13) are first converted into the following system of first order equations. For this purpose we introduce new dependent variables $u(x,y)$, $v(x,y)$, $p(x,y)$ and $g(x,y)$ so that the transformed momentum and energy equations can be written as

$$f' = u \tag{16}$$

$$u' = v \tag{17}$$

$$g' = p \tag{18}$$

$$P_1 v' + P_2 f u' - u^2 - P_3 g u' + P_4 g - P_5 u = x\left(u\frac{\partial u}{\partial x} - v\frac{\partial f}{\partial x}\right) \tag{19}$$

$$\frac{1}{\Pr} p' + P_1 f\, p - P_4 u\, g + P_6 \left(u'\right)^2 + P_7 u^2 = x\left(u\frac{\partial g}{\partial x} - p\frac{\partial f}{\partial x}\right) \tag{20}$$

where $x = x, \theta = g$ and $P_1 = \dfrac{1}{1+\varepsilon\theta}$, $P_2 = 1 + \dfrac{x\cos x}{\sin x}$, $P_3 = \dfrac{\varepsilon}{\left(1+\varepsilon\theta\right)^2}$,
$P_4 = \dfrac{\sin x}{x}$, $P_5 = M$, $P_6 = N\xi^2$, $P_7 = J$

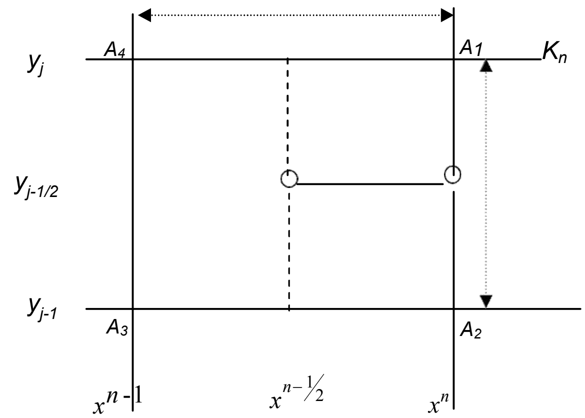


Fig. 2. Net rectangle of difference approximations for the box scheme.

and the boundary conditions (13) are

$$\begin{aligned} &f(x,0)=0, \quad u(x,0)=v(x,0)=0,\ g(x,0)=1,\ x\ \rangle 0 \\ &u(0,y)=v(0,y)=0, \quad g(0,y)=1 \\ &u\to 0, v\to 0\ as\ y\to\infty,\ x\ \rangle 0 \end{aligned} \tag{21}$$

Now we consider the net rectangle on the (x, y) plane as shown in the Fig. 2 and denote the net points by

$$x^0=0,\ x^n=x^{n-1}+k_n,\ n=1,2,3,—,N\ and\ y_0=0,\ y_j=y_{j-1}+h_j,\ j=1,2,3,—,J \tag{22}$$

Here n and j are just sequence of numbers on the (x, y) plane, k_n and h_j are the variable mesh widths. Approximate the quantities *f*, *u*, *v* and p , at the points (x^n, y_j) of the net by $f_j^n,\ u_j^n,\ v_j^n,\ p_j^n$ which call net function. It is also employed that the notation P_j^n for the quantities midways between net points shown in Fig. 2 and for any net functions as

$$x^{n-\frac{1}{2}}=\frac{1}{2}\left(x^n+x^{n-1}\right) \tag{23}$$

$$y_{j-\frac{1}{2}}=\frac{1}{2}\left(y_j+y_{j-\frac{1}{2}}\right) \tag{24}$$

$$g_j^{n-\frac{1}{2}}=\frac{1}{2}\left(g_j^n+g_j^{n-1}\right) \tag{25}$$

The finite difference approximation according to box method to the three first order ordinary differential equation (18)-(20) are written for the mid - point $\left(x^n, y_{j-\frac{1}{2}}\right)$ of the segment A_1A_2 shown in Fig. 2.

$$\frac{f_j^n - f_{j-1}^n}{h_j} = u_{j-1/2}^n = \frac{u_{j-1}^n + u_j^n}{2} \tag{26}$$

$$\frac{u_j^n - u_{j-1}^n}{h_j} = v_{j-1/2}^n = \frac{v_j^n + v_{j-1}^n}{2} \tag{27}$$

$$\frac{g_j^n - g_{j-1}^n}{h_j} = p_{j-1/2}^n = \frac{p_j^n + p_{j-1}^n}{2} \tag{28}$$

The finite difference approximation to the first order differential equation (19) and (20) are written for the mid - point $\left(x^{n-1/2}, y_{j-1/2}\right)$ of the rectangle $A_1 A_2 A_3 A_4$. This procedure yields

$$\begin{aligned}&\frac{1}{2}\left(P_1\frac{v_j^n - v_{j-1}^n}{h_j}\right) + \frac{1}{2}\left(P_1\frac{v_j^{n-1} - v_{j-1}^{n-1}}{h_j}\right) + \left(P_2 f\, v\right)_{j-1/2}^{n-1/2} - \left(u^2\right)_{j-1/2}^{n-1/2} - \left(P_3 g v\right)_{j-1/2}^{n-1/2} + \left(P_4 g\right)_{j-1/2}^{n-1/2} \\ &-\left(P_5\, u\right)_{J-1/2}^{n-1/2} = x_{j-1/2}^{n-1/2}\left(u_{j-1/2}^{n-1/2}\frac{u_{j-1/2}^n - u_{j-1/2}^{n-1}}{k_n} - v_{j-1/2}^{n-1/2}\frac{f_{j-1/2}^n - f_{j-1/2}^{n-1}}{k_n}\right)\end{aligned} \tag{29}$$

$$\begin{aligned}&\frac{1}{2\Pr}\left(\frac{p_j^n - p_{j-1}^n}{h_j}\right) + \frac{1}{2\Pr}\left(\frac{p_j^{n-1} - p_{j-1}^{n-1}}{h_j}\right) + \left(P_1 f\, p\right)_{j-1/2}^{n-1/2} - \left(P_4 u g\right)_{j-1/2}^{n-1/2} \\ &+\left(P_6\, v^2\right)_{j-1/2}^{n-1/2} + \left(P_7\, u^2\right)_{j-1/2}^{n-1/2} = x_{j-1/2}^{n-1/2}\left(u_{j-1/2}^{n-1/2}\frac{g_{j-1/2}^n - g_{j-1/2}^{n-1}}{k_n} - p_{j-1/2}^{n-1/2}\frac{f_{j-1/2}^n - f_{j-1/2}^{n-1}}{k_n}\right)\end{aligned} \tag{30}$$

The above equations are to be linearized by using Newton's Quasi-linearization method. The program used was developed with Lahey Fortran 90. Then linear algebraic equations can be written in block matrix which form a coefficient matrix. The whole procedure, namely reduction to first order followed by central difference approximations, Newton's Quasi-linearization method and the block Thomas algorithm, is well known as the Keller-box method.

5 Results and Discussion

For getting appropriate solutions and proper curves by using suitable data for some test values of Prandtl's number *Pr* = 0.10, 0.72, 1.00, 1.74 and for a wide range of the values of the viscous dissipation parameter *N* = 0.90, 0.70, 0.50, 0.20, temperature dependent viscosity parameter ε = 1.50, 1.20, 1.00, 0.70, 0.40, 0.20, magnetic parameter M = 1.50, 1.20, 0.90, 0.60, 0.30 and the Joule heating parameter *J* = 0.90, 0.70, 0.30, 0.10. First demarcate the area of a sphere and also satisfy its boundary conditions. Then for worthy vaIues of various paprameters the programe is convergent so that it gives suitable data. we know the values of the functions $f(x,y)$, $\theta(x,y)$ and their derivatives for the different values of the Prandtl's number *Pr*, the temperature dependent parameter ε, the viscous dissipation parameter *N*, the magnetic parameter *M* and the Joule heating parameter *J*. Also we may calculate the numerical values of the heat transfer coefficient $\theta'(x,0)$ and the velocity gradient $f''(x,0)$ on the surface that are important from the physical point of view.

Fig. 3 (a) and Fig. 3 (b) deal with the effect of the viscous dissipation parameter *N* (= 0.20, 0.50, 0.70, 0.90) for distinct values of the controlling parameters *Pr* = 0.72, *J* = 0.80, *M* = 0.50 and ε = 2.00 on the velocity profiles $f'(x,y)$ and the temperature profiles $\theta(x,y)$. From Fig. 3(a), it is revealed that the velocity profile $f'(x,y)$ increases very small with the increase of the viscous dissipation parameter *N* which indicates that viscous dissipation increases the fluid motion slowly. In Fig. 3 (b) it is shown that the temperature profiles $\theta(x,y)$ small increase for increasing values of *N* with other controlling parameter.

The effects of magnetic parameter or Hartmann number *M* for *Pr* = 0.72, *J* = 0.70, *N* = 0.50 and ε = 2.00 on the velocity profiles and temperature profiles are shown in Figs. 4(a) to 4(b). Fig. 4(a) and Fig. 4(b) represent respectively the effects of magnetic parameter *M* on the velocity and temperature profiles. From these figures, it is seen that the velocity profiles decrease with the increasing values of *M* and the temperature profiles increase with the increasing values of *M*.

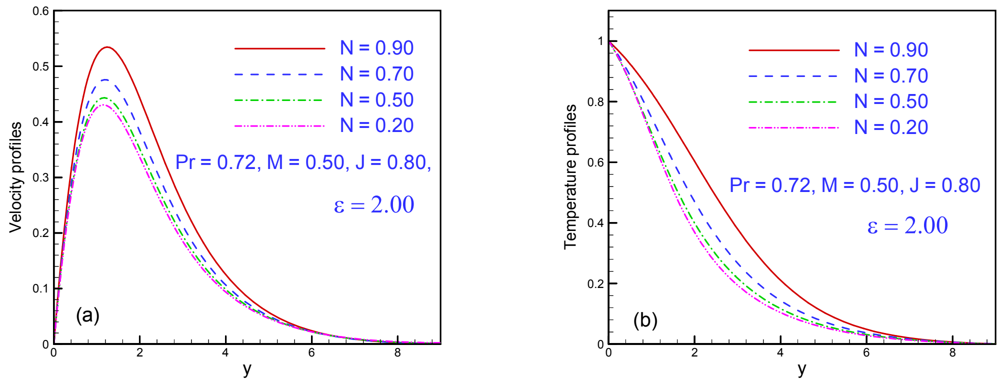
From Fig. 5(a) shows the effects of the velocity profile for various values of the temperature dependent parameter ε = 0.20, 0.40, 0.70, 1.00, 1.20, 1.50 while the other controlling parameters *Pr* = 0.72, *J* = 0.90, *M* = 1.00 and *N* = 0.60. Corresponding distribution of the temperature profile is shown in Fig. 5(b). From Fig. 5(a), it is seen that if the temperature dependent viscosity parameter increases, the velocity of the fluid also increases. On the other hand, it is observed that the temperature profile increases within the boundary layer due to increase of temperature dependent viscosity parameter ε which is evident from Fig. 5(b).

From Fig. 6(a) that an increase in the Joule heating parameter is associated with a considerable increase in velocity profiles but near the surface of the plate the velocity increases and become maximum and then decreases and finally approaches to zero. Fig. 6(b) shows the distribution of the temperature profiles $\theta(x,y)$ against y for the same values of the Joule heating parameter *J* and each of which attains maximum at the surface.

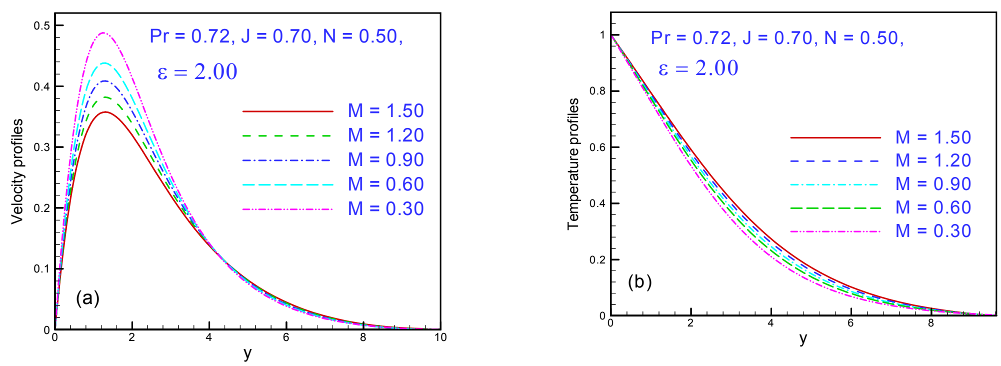
Fig. 7(a) depicts the velocity profile for different values of the Prandtl's number, *Pr* = 0.10, 0.72, 1.00, 1.74 while the others controlling parameters *N* = 0.80, *J* = 0.50, *M* = 1.00 and ε = 2.00. Corresponding distribution of the temperature profile $\theta(x,y)$ in the fluids is shown in Fig. 7(b). From Fig. 7(a), it is illustrate that if the Prandtl's number increases the velocity of the fluid decreases. On the other hand, from Fig. 7(b) it is seen that the temperature profile decreases within the boundary layer due to increase of the Prandtl number *Pr*.

Numerical values of the velocity gradient $f''(x,0)$ and the heat transfer coefficient $\theta'(x,0)$ are depicted graphically in Fig. 8(a) and Fig. 8(b) respectively against the axial distance x for different values of the viscous dissipation parameter *N* (= 0.20, 0.50, 0.70, 0.90) for the fluid having Prandtl's number *Pr* = 0.72, *J* = 0.80, M = 1.00 and ε = 2.00. From Fig. 8(a), it is performed that the skin-friction $f''(x,0)$ increases when the viscous dissipation parameter *N* increases. It is also observed in Fig. 8(b), the heat transfer coefficient $\theta'(x,0)$ decreases while viscous dissipation parameter *N* increases.

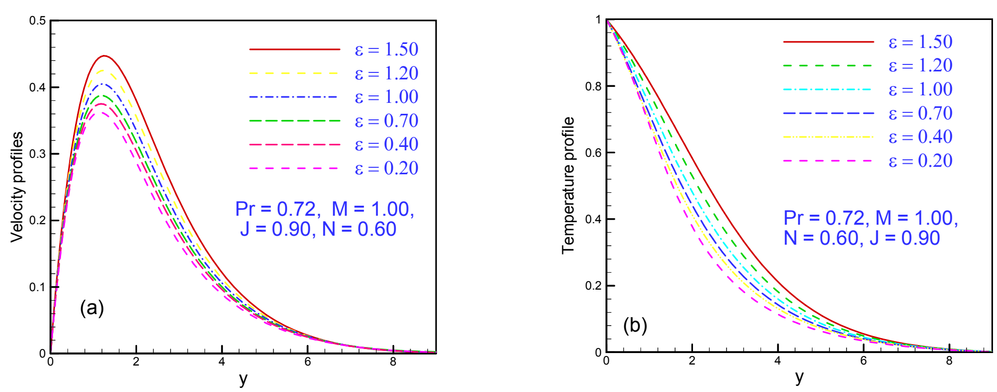
The effects of magnetic parameter or Hartmann number *M* for various values of *Pr* = 0.72, *J* = 0.70, *N* = 0.50 and ε = 2.00 on the local skin friction coefficient $f''(x,0)$ and the heat transfer coefficient $\theta'(x,0)$ are shown in Fig. 9(a) and Fig. 9(b). From Fig. 9(a) and Fig. 9(b) it is evident that the increasing values of magnetic parameter *M* leads to decrease both the skin friction co-efficient $f''(x,0)$ and the heat transfer co-efficient $\theta'(x,0)$.



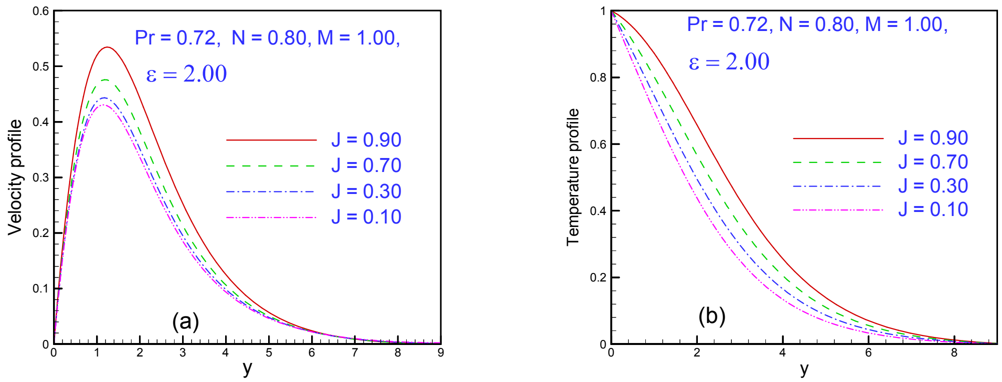
3(a) and 3(b). Variation of dimensionless velocity profiles $f'(x, y)$ and temperature profiles $\theta(x, y)$ against dimensionless distance y for different values of viscous dissipation parameter N with Pr = 0.72, ε = 2.00, M = 1.00 and J = 0.80



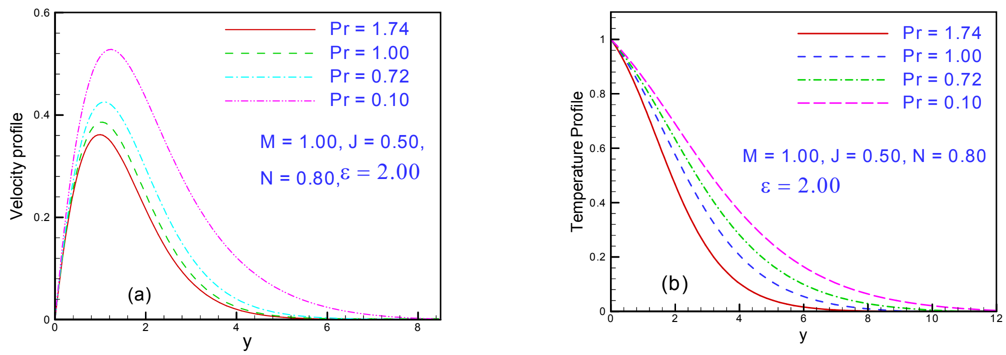
4(a) and 4(b). Variation of dimensionless velocity profiles $f'(x, y)$ and temperature profiles $\theta(x, y)$ against dimensionless distance y for various values of magnetic parameter M with Pr = 0.72, ε = 2.00, J = 0.70 and N= 0.50.



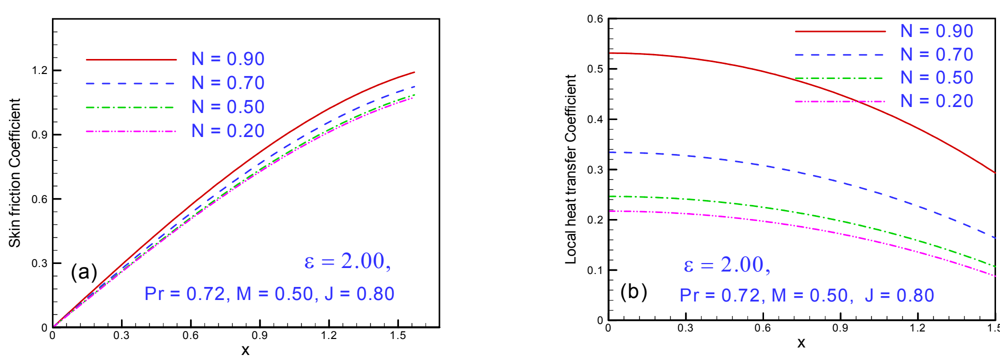
5(a) and 5(b). Variation of dimensionless velocity profiles $f'(x, y)$ and temperature profiles $\theta(x, y)$ against dimensionless distance y for distinct values of variation viscosity parameter ε with Pr = 0.72, N = 0.60, M = 1.00 and J = 0.90



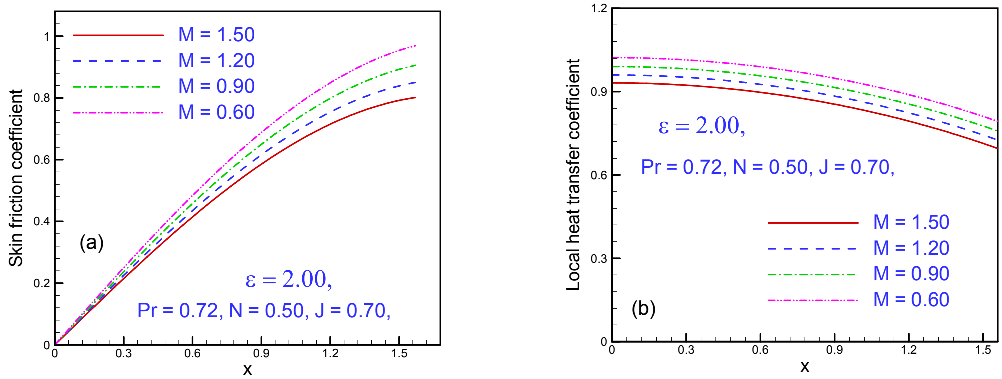
6(a) and 6(b). Variation of dimensionless velocity profiles $f'(x,y)$ and temperature profiles $\theta(x,y)$ against dimensionless distance y for different values of Joule heating parameter *J* with *N* = 0.80, *M* = 1.00, ε = 2.00 and *J* = 0.60



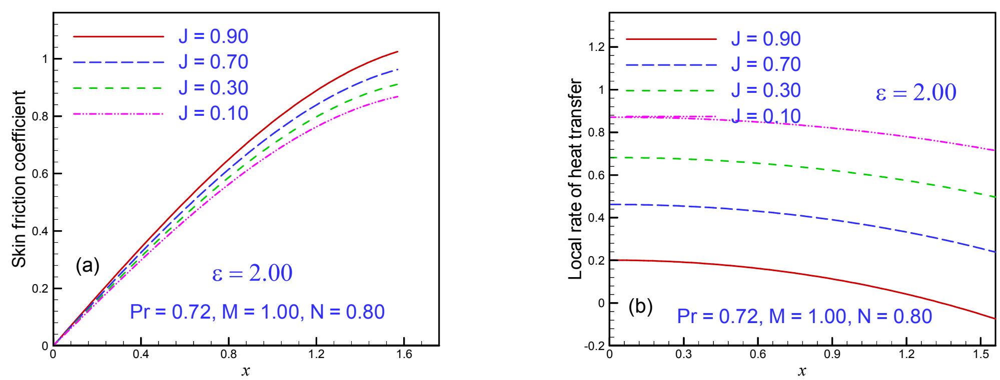
7(a) and 7(b). Variation of velocity profiles $f'(x,y)$ and temperature profiles $\theta(x,y)$ against dimensionless distance y for various values of Prandtl's number Pr with *N* = 0.80, *J* = 0.50, M= 1.00 and ε = 2.00



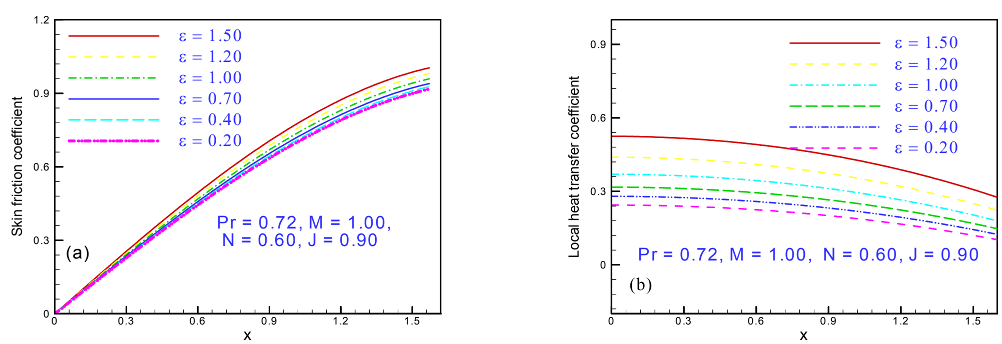
8(a) and 8(b). Variation of dimensionless skin friction coefficient $f''(x,0)$ and the heat transfer coefficient $\theta'(x,0)$ against dimensionless distance x for distinct values of viscous dissipation parameter N with Pr = 0.72, ε = 2.00, J = 0.80 and M = 1.00



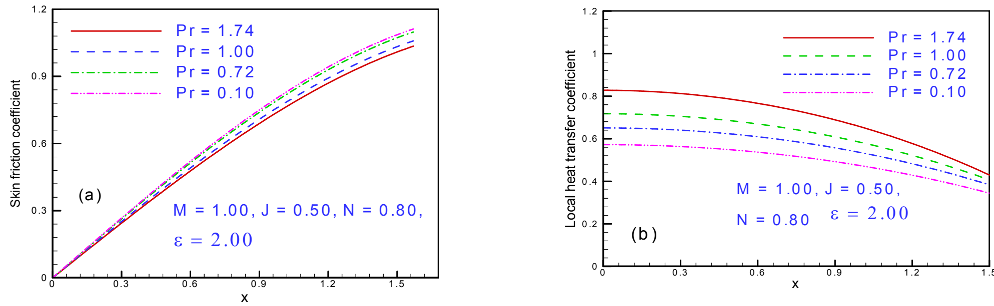
9(a) and 9(b). Variation of dimensionless skin friction coefficient $f''(x,0)$ and the heat transfer coefficient $\theta'(x,0)$ against dimensionless distance x for different values of magnetic parameter M with Pr = 0.72, J = 0.70, N = 0.50 and ε = 2.00



10(a) and 10(b). Variation of dimensionless skin friction coefficient $f''(x,0)$ and the heat transfer coefficient $\theta'(x,0)$ against dimensionless distance x for distinct values of Joule heating parameter J with N = 0.80, N= 0.80, M = 1.00 and ε = 2.00



11(a) and 11(b). Variation of dimensionless skin friction coefficient $f''(x,0)$ and the heat transfer coefficient $\theta'(x,0)$ against dimensionless distance x for various values of variation viscosity parameter ε with N = 0.60, J = 0.90, M = 1.00 and Pr = 0.72



12(a) and 12(b). Variation of dimensionless skin friction coefficient $f''(x,0)$ and the heat transfer coefficient $\theta'(x,0)$ against dimensionless distance x for distinct values of Prandtl's number *Pr* with *N* = 0.80, *J* = 0.50, M = 1.00 and ε = 2.00

The analysis of the effects of Joule heating parameter *J* (= 0.10, 0.30, 0.70, 0.90) on the surface shear stress in terms of local skin-friction coefficient $f''(x,0)$ and the rate of heat transfer in terms of the Nusselt number $\theta'(x,0)$ against x for *Pr* = 0.72, *N* = 0.80, *M* = 1,00 and ε = 2.00 is shown in Fig. 10(a) and Fig. 10(b). It is found that the values of the skin-friction $f''(x,0)$ and the Nusselt number $\theta'(x,0)$ both increases for increasing values of Joule heating parameter *J*. Here it has been observed that the values of the skin-friction $f''(x,0)$ increases by 76.734% and the Nusselt number $\theta'(x,0)$ increases by 83.264% while *J* increased from 0.10 to 0.90.

From Fig. 11(a), it is found that increase in the value of the temperature dependent viscosity parameter ε leads to increase of the value of the shear stress coefficient $f''(x,0)$ which is usually expected. Again from Fig. 11(b) it is illustrated that the increase of the temperature dependent viscosity parameter ε leads to increase of the heat transfer coefficient $\theta'(x,0)$.

From Fig. 12(a), it is seen that increase in the value of the Prandtl's number *Pr* (= 0.10, 0.72, 1.00, 1.74) leads to decrease of the value of shear stress $f''(x,0)$. Similar results hold in the Nusselt number $\theta'(x,0)$ shown in Fig. 12(b) for the same values of Prandtl number *Pr* while *J* = 0.50, *N* = 0.80, M = 1.00 and ε = 2.00.

Table 1. Skin friction coefficient and Nusselt number against *x* for different values of magnetic parameter *M* with other controlling parameters *Pr* = 0.72, *J* = 0.70, *N* =0.50 and ε = 2.00

	***M* = 0.60**		***M* = 0.90**		***M* = 1.20**		***M* = 1.50**	
x	$f''(x,0)$	$\theta'(x,0)$	$f''(x,0)$	$\theta'(x,0)$	$f''(x,0)$	$\theta'(x,0)$	$f''(x,0)$	$\theta'(x,0)$
0.00000	0.00000	1.02248	0.00000	0.99028	0.00000	0.96003	0.00000	0.93160
0.10472	0.08795	1.02141	0.08340	0.98919	0.07933	0.95893	0.07569	0.93045
0.20944	0.17522	1.01838	0.16613	0.98612	0.15800	0.95582	0.15073	0.92731
0.40143	0.33132	1.00768	0.31395	0.97528	0.29842	0.94485	0.28455	0.91625
0.50615	0.41318	0.99902	0.39132	0.96651	0.37181	0.93597	0.35437	0.90730
0.80285	0.62615	0.96344	0.59178	0.93047	0.56121	0.89952	0.53401	0.87053
1.01229	0.75407	0.92818	0.71112	0.89478	0.67310	0.86343	0.63938	0.83415
1.20428	0.85072	0.88812	0.80021	0.85425	0.75569	0.82249	0.71638	0.79291
1.30900	0.89399	0.86301	0.83950	0.82887	0.79161	0.79685	0.74945	0.76711
1.50098	0.95406	0.81068	0.89264	0.77603	0.83900	0.74355	0.79206	0.71350
1.57080	0.96930	0.78954	0.90550	0.75471	0.84993	0.72206	0.80142	0.69192

The values of the local skin friction coefficient $f''(x,0)$ and the heat transfer coefficient $\theta'(x,0)$ for different values of magnetic parameter or Hartmann number *M* while *Pr* = 0.72, *N* = 0.50, *J* = 0.70 and ε = 2.00 are given in Table 1 which is shown in below. Moreover, it is examined that the values of local skin friction coefficient $f''(x,0)$ decrease at various position of x for magnetic parameter *M* = 0.60, 0.90, 1.20, 1.50. The rate of the local skin friction coefficient $f''(x,0)$ decreases by 14.72% as the magnetic parameter *M* changes from 0.60 to 1.50 and x = 0.80285. Furthermore, it is seen that the numerical values of the heat transfer coefficient $\theta'(x,0)$ decrease as magnetic parameter or Hartmann number *M* rising. The rate of decrease of the heat transfer coefficient $\theta'(x,0)$ is 9.64% at position x = 0.80285 as the magnetic parameter *M* changes from 0.60 to 1.50.

Table 2. Comparisons of the present numerical results of Nu_x or the Prandtl numbers *Pr* = 0.70, 7.00 without effect of the viscous dissipation parameter, Joule heating parameter and temperature dependent viscosity parameter with those obtained by Molla et al. [18] and Nazar et al. [17]

	***Pr* = 0.70**			***Pr* = 7.00**		
x in degree	**Naza et al. [17]**	**Molla et al. [18]**	**Present**	**Naza et al. [17]**	**Molla et al. [18]**	**Present**
0	0.4576	0.4576	0.4492	0.9595	0.9582	0.9527
10	0.4565	0.4564	0.4485	0.9572	0.9558	0.9487
20	0.4533	0.4532	0.4467	0.9506	0.9492	0.9410
30	0.4480	0.4479	0.4401	0.9397	0.9383	0.9279
40	0.4405	0.4404	0.4340	0.9239	0.9231	0.9140
50	0.4308	0.4307	0.4259	0.9045	0.9034	0.8945
60	0 .4189	0.4188	0.4106	0.8801	0.8791	0.8611
70	0.4046	0.4045	0.3982	0.8510	0.8501	0.8395
80	0.3879	0.3877	0.3769	0.8168	0.8161	0.8133
90	0.3684	0.3683	0.3607	0.7774	0.7768	0.7707

The comparison of the heat transfer coefficient between the present work and the work of Nazar et.al. and Molla et.al. are presented in Table 2 respectively. We observed that the comparison without the effect of viscous dissipation parameter, temperature dependent viscosity parameter, magnetic parameter and Joule heating parameter in the present problem is similar to the previous work.

6 Conclusions

From the present investigation, the following conclusions may be drawn:

- Increase in the values of viscous dissipation parameter *N* leads to an increase in the velocity profile, the temperature profile, the local skin friction coefficient $f''(x,0)$ but the rate of heat transfer $\theta'(x,0)$ decreases with the increase in viscous dissipation parameter *N* for *J* =0.80, ε = 2.00, *M* = 1.00 and *Pr* = 0.72.
- The velocity profile, the skin friction coefficient $f''(x,0)$ and the rate of heat transfer $\theta'(x,0)$ all are decreasing for increasing values of the magnetic parameter or Hartmann number *M*, but the temperature profile increase with the increase of magnetic parameter *M*.
- The velocity profiles, the temperature profiles, the local skin friction coefficient $f''(x,0)$ and also the heat transfer coefficient $\theta'(x,0)$ increase significantly when the values of variable viscosity parameter ε increase.

- Significant effects of Joule heating parameter J on velocity and temperature profiles as well as on local skin friction coefficient and the rate of heat transfer have been found in this investigation. An increase in the values of Joule heating parameter J leads to an increase in both velocity and temperature profiles. The local skin friction coefficient $f''(x,0)$ increases at a different position of x, also the rate of heat transfer $\theta'(x,0)$ increases at different position of x for Pr =0.72, ε = 2.00 M = 1.00and N = 0.80.
- Increasing values of Prandtl's number Pr leads to decrease the velocity profiles. The temperature profiles, the local skin friction coefficient $f''(x,0)$ and also the local rate of heat transfer $\theta'(x,0)$ increases with the increase of Prandtl's number Pr when J = 0.50, ε = 2.00, M = 1.00 and N = 0.80.

Competing Interests

Authors have declared that no competing interests exist.

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