

A New Odd Lindley-Gompertz Distribution: Its Properties and Applications

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

This article presents a comprehensive study of an odd Lindley-Gompertz distribution which has already been proposed in the literature but without any properties. The present study unlike the previous one has considered the derivation of several properties of the odd Lindley-Gompertz distribution with their graphical representations and discussions which has not been done in the first proposition of the distribution. The study looks at properties such as survival (or reliability) function, the hazard function, the cumulative hazard function, the reverse hazard function, the odds function, quantile function, moments, moment generating function, characteristic function, cumulant generating function, distribution of order statistics and maximum likelihood estimation of the distribution's parameters none of which was treated by the previous author of the model. An illustration to evaluate the goodness-of-fit of the odd Lindley-Gompertz distribution has also been done using two real life datasets and the results show that the model fits the datasets better than the five other distributions considered in this present study.

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1 Introduction

The Gompertz distribution is both skewed to the right and to the left. It is a generalization of the exponential distribution and is commonly used in many applied problems, particularly in lifetime data analysis [1]. The Gompertz distribution has been applied in the analysis of survival, in some sciences such as gerontology [2], computer science [3], biological science [4], and marketing science [5]. The hazard rate function of the Gompertz distribution is an increasing function and often applied to describe the distribution of adult life spans by actuaries and demographers [6].

Several methods or families of distributions have been proposed for adding parameters to all forms of probability distributions which makes the resulting distribution better than the standard counterparts in modeling skewed datasets. A short list of these recently proposed families of distributions include a Lomax-G family by [7], a new generalized Weibull-G family by [8], a Beta Marshall-Olkin family of distributions by [9], Logistic-X by [10], a new Weibull-G family by [11], a Lindley-G family by [12], a Gompertz-G family by [13] and Odd Lindley-G family by [14] etc. Meanwhile a comprehensive list of all these families can be found in [15].

Based on these families of probability distribution, many studies have proposed different extensions of the Gompertz distribution and some of the recent and known studies include the generalized Gompertz distribution by [16] which was based on an idea of [17], the Beta Gompertz distribution by [18], the odd generalized Exponential-Gompertz distribution by [19], the Transmuted Gompertz distribution by [20], the Lomax-Gompertz distribution by [21] and the odd Lindley-Gompertz distribution by [22].

Kuje et al. [22] used the family of distributions by [14] to propose an odd Lindley-Gompertz distribution due to fact that the odd Lindley-Weibull distribution which was based on the odd Lindley-G family was found to fit real dataset much better than other extensions of the Weibull distribution such as exponentiated Weibull distribution, beta Weibull distribution, Kumaraswamy Weibull distribution and the conventional Weibull distribution [14].

This article is an improvement over the work of [22] who did not capture any of the properties of the odd Lindley-Gompertz distribution which are very useful in engineering and medicine. Hence, our interest in this article is to develop a new odd Lindley-Gompertz distribution using the odd Lindley-G family of probability distributions proposed by [14] with many properties such as survival (or reliability) function, the hazard (or failure rate) function, the cumulative hazard function, the reverse hazard function, the odds function, quantile function, moments, moment generating function, characteristic function, cumulant generating function, distribution of order statistics and maximum likelihood estimation of the distribution's parameters all of which are not found in [22].

The cumulative distribution function (cdf) and probability density function (pdf) of the Gompertz distribution with parameters θ and λ are given as:

$$G(x) = 1 - e^{-\frac{\theta}{\lambda}(e^{\lambda x} - 1)} \quad (1)$$

and

$$g(x) = \theta e^{\lambda x} e^{-\frac{\theta}{\lambda}(e^{\lambda x} - 1)} \quad (2)$$

respectively. For $x \geq 0, \theta > 0, \lambda > 0$ where θ and λ are scale and shape parameters of the model respectively.

The rest of this article is organized in sections as follows: the new model with a graphical representation of its pdf, cdf, survival and hazard functions are given in section 2. Section 3 derived some properties of the new odd Lindley-Gompertz distribution. Section 4 presents the estimation of unknown parameters of the distribution using maximum likelihood estimation. An application of the new model to two real life datasets is done in section 5 with a brief summary and conclusion in section 6.

2 New Odd Lindley-Gompertz Distribution (OLinGomD)

According to [14], the cumulative distribution function (cdf) and the probability density function (pdf) of the Odd Lindley-G family of distributions are defined as:

$$F(x) = \int_{-\infty}^{\frac{G(x)}{1-G(x)}} \frac{\alpha^2}{\alpha+1} (1+t) e^{-\alpha t} dt = 1 - \frac{\alpha + (1-G(x))}{(1+\alpha)(1-G(x))} \exp \left\{ -\alpha \left[\frac{G(x)}{1-G(x)} \right] \right\} \quad (3)$$

and

$$f(x) = \frac{\alpha^2 g(x)}{(1+\alpha)(1-G(x))^3} \exp \left\{ -\alpha \left[\frac{G(x)}{1-G(x)} \right] \right\} \quad (4)$$

respectively, where $g(x)$ and $G(x)$ are the pdf and the cdf of any continuous distribution to be modified respectively and $\alpha > 0$ is the shape parameter of the family responsible for additional skewness and flexibility in the modified model.

Substituting equation (1) and (2) in (3) and (4) above and simplifying, we obtain the cdf and pdf of the new OLinGomD for a random variable X as follows:

$$F(x) = 1 - \frac{\alpha + e^{-\frac{\theta}{\lambda}(e^{\lambda x}-1)}}{(1+\alpha) e^{-\frac{\theta}{\lambda}(e^{\lambda x}-1)}} e^{\alpha \left[1 - e^{-\frac{\theta}{\lambda}(e^{\lambda x}-1)} \right]} \quad (5)$$

and

$$f(x) = \frac{\alpha^2 \theta e^{\lambda x} e^{-\frac{\theta}{\lambda}(e^{\lambda x}-1)}}{(1+\alpha)} e^{\alpha \left[1 - e^{-\frac{\theta}{\lambda}(e^{\lambda x}-1)} \right]} \quad (6)$$

“respectively, where, $x > 0, \alpha > 0, \theta > 0$, α and λ are the shape parameters and θ is the scale parameter”.

The survival (or reliability) function, the hazard (or failure rate) function, the cumulative hazard function, the reverse hazard function and the odds function for the OLinGomD can be derived by substituting equation (5) and (6) above and simplifying to obtain (7), (8), (9), (10) and (11) respectively as follows:

$$S(x) = 1 - F(x) = \frac{\alpha + e^{-\frac{\theta}{\lambda}(e^{\lambda x}-1)}}{(1+\alpha) e^{-\frac{\theta}{\lambda}(e^{\lambda x}-1)}} e^{\alpha \left[1 - e^{-\frac{\theta}{\lambda}(e^{\lambda x}-1)} \right]} \quad (7)$$

$$h(x) = \frac{f(x)}{S(x)} = \frac{\alpha^2 \theta e^{\lambda x} e^{\frac{\theta}{\lambda}(e^{\lambda x}-1)}}{\alpha + e^{-\frac{\theta}{\lambda}(e^{\lambda x}-1)}} \tag{8}$$

$$H(x) = -\ln S(x) = -\log \left\{ \frac{\alpha + e^{-\frac{\theta}{\lambda}(e^{\lambda x}-1)} e^{\alpha \left[1 - e^{-\frac{\theta}{\lambda}(e^{\lambda x}-1)} \right]}}{(1 + \alpha) e^{-\frac{\theta}{\lambda}(e^{\lambda x}-1)}} \right\} \tag{9}$$

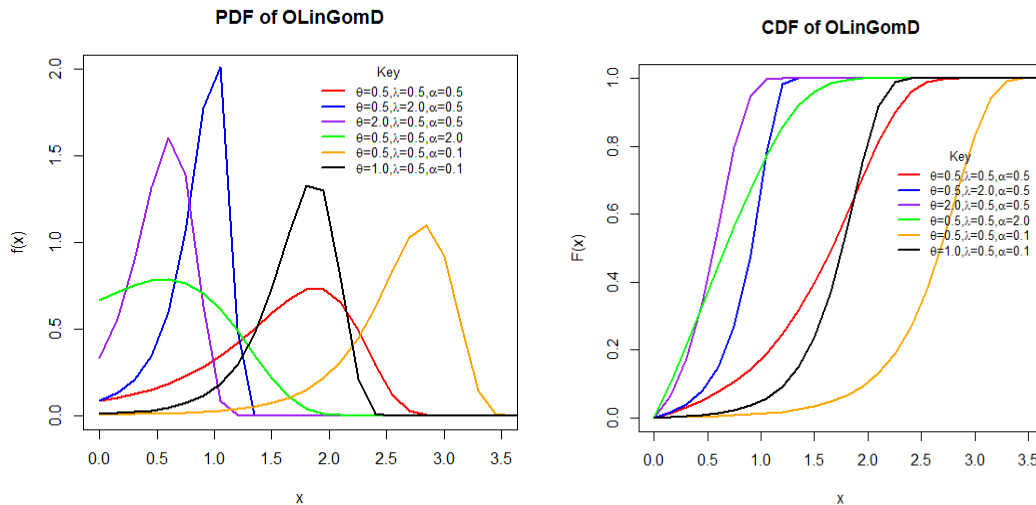
$$Rh(x) = \frac{f(x)}{F(x)} = \frac{\alpha^2 \theta e^{\lambda x} e^{\frac{\theta}{\lambda}(e^{\lambda x}-1)} e^{\alpha \left[1 - e^{-\frac{\theta}{\lambda}(e^{\lambda x}-1)} \right]}}{(1 + \alpha) e^{-\frac{\theta}{\lambda}(e^{\lambda x}-1)} - \left(\alpha + e^{-\frac{\theta}{\lambda}(e^{\lambda x}-1)} \right) e^{\alpha \left[1 - e^{-\frac{\theta}{\lambda}(e^{\lambda x}-1)} \right]}} \tag{10}$$

and

$$O(x) = \frac{F(x)}{S(x)} = \frac{(1 + \alpha) e^{-\frac{\theta}{\lambda}(e^{\lambda x}-1)}}{\left(\alpha + e^{-\frac{\theta}{\lambda}(e^{\lambda x}-1)} \right) e^{\alpha \left[1 - e^{-\frac{\theta}{\lambda}(e^{\lambda x}-1)} \right]}} - 1, \tag{11}$$

respectively, where $x > 0, \alpha, \beta, \theta > 0$ and α and λ are the shape parameters while θ is the scale parameter.

Graphical representation of Pdf, Cdf, survival and hazard functions of OLinGomD: The plots of pdf, cdf, survival and hazard functions of the OLinGomD using some parameter values are displayed in the figures below.



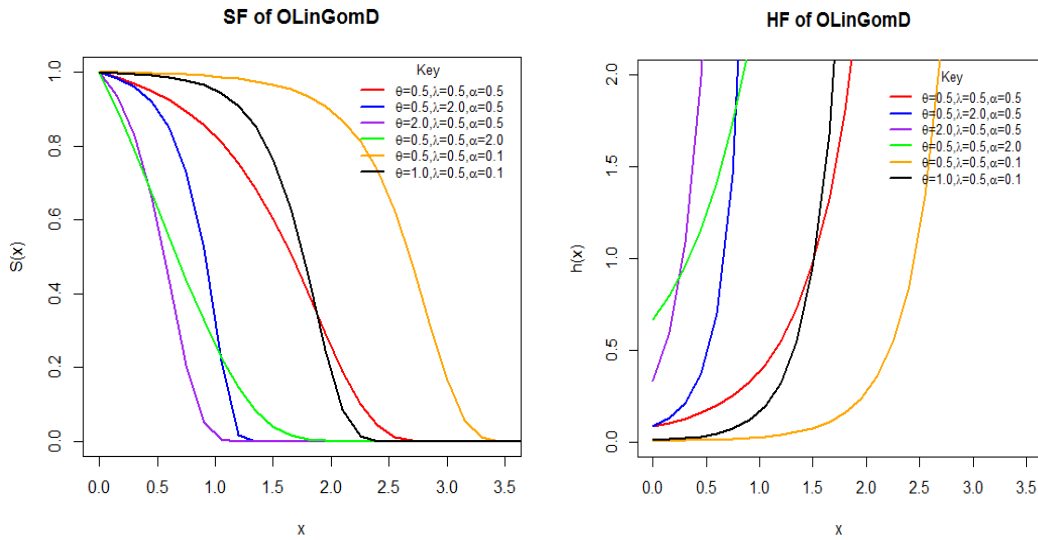


Fig. 1. Plots of the PDF, CDF, Survival function (SF), Hazard function (HF) of the OLinGomD for selected parameter values

From the plots in Fig. 1 above, it shows that the pdf of the OLinGomD is skewed with various shapes and therefore will be a good model for different kinds of datasets. It is also found that the plots are in line with the standard limiting behavior of the pdf and cdf. The plot of the survival function in the figure above also shows that the probability of survival equals one at initial time and it decreases as time increases and equals zero as it approaches infinity. Also worthy to note is the fact that the hazard function increases as time increases. This means that the OLinGomD could be appropriate for modeling time dependent events, where risk or hazard increases with time or age.

3 Mathematical and Statistical Properties of OLinGomD

3.1 Moments

Let X denote a continuous random variable, the n^{th} moment of X is given by:

$$\mu'_n = E(X^n) = \int_0^\infty x^n f(x) dx \tag{12}$$

where $f(x)$ the pdf of the OLinGomD is as given in equation (6)

Before substitution in (12), we perform the expansion and simplification of (6) as follows:

First, by expanding the exponential term in (6) using power series, we obtain:

$$e^{\alpha \left[1 - e^{-\frac{\theta}{\lambda} (e^{\lambda x} - 1)} \right]} = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} \left(1 - e^{-\frac{\theta}{\lambda} (e^{\lambda x} - 1)} \right)^k \tag{13}$$

Making use of the result in (13) above, equation (6) becomes

$$f(x) = \sum_{k=1}^{\infty} \frac{\alpha^{2+k} \theta}{k!(1+\alpha)} e^{\lambda x} e^{2\frac{\theta}{\lambda}(e^{\lambda x}-1)} \left[1 - e^{\frac{\theta}{\lambda}(e^{\lambda x}-1)} \right]^k \quad (14)$$

Also, using the generalized binomial theorem, we can write the last term from the above result as:

$$\left[1 - e^{\frac{\theta}{\lambda}(e^{\lambda x}-1)} \right]^k = \sum_{l=0}^{\infty} (-1)^l \binom{k}{l} e^{\frac{\theta}{\lambda}(e^{\lambda x}-1)l} \quad (15)$$

Making use of the result in (15) above in equation (14) and simplifying, we obtain:

$$f(x) = \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^l \alpha^{2+k} \theta}{k!(1+\alpha)} \binom{k}{l} e^{-\frac{\theta}{\lambda}(l+2)x} e^{\lambda x} e^{\frac{\theta}{\lambda}(l+2)e^{\lambda x}} \quad (16)$$

Again, using series expansion of $e^{\frac{\theta}{\lambda}(l+2)e^{\lambda x}}$, we get:

$$f(x) = \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^l \alpha^{2+k} \theta (\theta(l+2))^m}{k!(1+\alpha) \lambda^m e^{\frac{\theta}{\lambda}(l+2)x}} \binom{k}{l} e^{\lambda(m+1)x} \quad (17)$$

Now, using the simplified pdf of the OLinGomD in equation (17), the n^{th} ordinary moment of the OLinGomD is derived as follows:

$$\mu'_n = E(X^n) = \int_0^{\infty} x^n f(x) dx = \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^l \alpha^{2+k} \theta (\theta(l+2))^m}{k!(1+\alpha) \lambda^m e^{\frac{\theta}{\lambda}(l+2)x}} \binom{k}{l} \int_0^{\infty} x^n e^{\lambda(m+1)x} dx \quad (18)$$

Making use of integration by substitution method in equation (18), we perform the following operations:

Let $-u = \lambda(m+1)x \Rightarrow x = -u [\lambda(m+1)]^{-1}$ which implies that $-\frac{du}{dx} = \lambda(m+1) \Rightarrow dx = -\frac{du}{\lambda(m+1)}$.

Substituting for x , and dx in equation (18) and simplifying; we have:

$$\mu'_n = E(X^n) = \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^l \alpha^{2+k} \theta (\theta(l+2))^m}{k!(1+\alpha) \lambda^m e^{\frac{\theta}{\lambda}(l+2)x}} \binom{k}{l} \left(-\frac{1}{\lambda(m+1)} \right)^{n+1} \int_0^{\infty} u^n e^{-u} du \quad (19)$$

Using the definition of complete Gamma function, we obtain the n^{th} ordinary moment of X for the OLinGomD as:

$$\mu'_n = E(X^n) = \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^l \alpha^{2+k} \theta (\theta(l+2))^m}{k!(1+\alpha) \lambda^m e^{\frac{\theta}{\lambda}(l+2)x}} \binom{k}{l} \frac{(-1)^{n+1} \Gamma(n+1)}{[\lambda(m+1)]^{n+1}} \quad (20)$$

The mean (μ_1'), variance (σ^2), coefficient of variation (CV), coefficient of skewness (CS), coefficient of kurtosis (CK), moment generating function (mgf) and characteristics function (cf) can all be calculated based on the ordinary moments in equation (20) using some simple and well-defined relationships.

The moment generating function of a random variable X can be obtained as

$$M_x(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x) dx \tag{21}$$

Applying power series expansion and simplifying equation (21) gives the following:

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^1 x^r f(x) dx = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) = \sum_{r=0}^{\infty} \frac{t^r}{r!} [\mu_r'] \tag{22}$$

Using the result in equation (22) and simplifying the integral in (21) therefore we have:

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \left[\sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^l \alpha^{2+k} \theta(\theta(l+2))^m}{k!(1+\alpha)\lambda^m e^{\frac{\theta}{\lambda}(l+2)}} \binom{k}{l} \frac{(-1)^{n+1} \Gamma(n+1)}{[\lambda(m+1)]^{n+1}} \right] \tag{23}$$

The characteristics function of a random variable X is defined by:

$$\varphi_x(t) = E(e^{itx}) = \int_0^{\infty} e^{itx} f(x) dx \tag{24}$$

Again, applying power series expansion and simplifying equation (24), we obtained the characteristics function of X as:

$$\varphi_x(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \left[\sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^l \alpha^{2+k} \theta(\theta(l+2))^m}{k!(1+\alpha)\lambda^m e^{\frac{\theta}{\lambda}(l+2)}} \binom{k}{l} \frac{(-1)^{r+1} \Gamma(r+1)}{[\lambda(m+1)]^{r+1}} \right] \tag{25}$$

The Cumulant generating function (CGF) is obtained as:

$$K_x(t) = \log \left\{ \sum_{r=0}^{\infty} \frac{t^r}{r!} \left[\sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^l \alpha^{2+k} \theta(\theta(l+2))^m}{k!(1+\alpha)\lambda^m e^{\frac{\theta}{\lambda}(l+2)}} \binom{k}{l} \frac{(-1)^{r+1} \Gamma(r+1)}{[\lambda(m+1)]^{r+1}} \right] \right\} \tag{26}$$

3.2 Quantile function

According to [23], the quantile function for any distribution is defined in the form $Q(u) = X_q = F^{-1}(u)$ where $Q(u)$ is the quantile function of $F(x)$ for $0 < u < 1$.

Taking $F(x)$ to be the *cdf* of the OLinGomD and inverting it as above will give us the quantile function as follows:

$$F(x) = 1 - \frac{\alpha + e^{-\frac{\theta}{\lambda}(e^{\lambda x} - 1)}}{(1 + \alpha) e^{-\frac{\theta}{\lambda}(e^{\lambda x} - 1)}} e^{\alpha \left[1 - e^{-\frac{\theta}{\lambda}(e^{\lambda x} - 1)} \right]} = u \tag{27}$$

Simplifying equation (27) above gives:

$$-(1 + \alpha)(1 - u)e^{-(\alpha + 1)} = -\frac{\alpha + e^{-\frac{\theta}{\lambda}(e^{\lambda x} - 1)}}{e^{-\frac{\theta}{\lambda}(e^{\lambda x} - 1)}} e^{\frac{\alpha + e^{-\frac{\theta}{\lambda}(e^{\lambda x} - 1)}}{e^{-\frac{\theta}{\lambda}(e^{\lambda x} - 1)}}} \tag{28}$$

In the expression above, it can be seen that $-\frac{\alpha + e^{-\frac{\theta}{\lambda}(e^{\lambda x} - 1)}}{e^{-\frac{\theta}{\lambda}(e^{\lambda x} - 1)}}$ is the Lambert function of the real argument,

$$-(1 + \alpha)(1 - u)e^{-(\alpha + 1)} \text{ since the Lambert function is defined as: } w(x)e^{w(x)} = x$$

Also note that the Lambert function has two branches with a branching point located at $(-e^{-1}, 1)$. The lower branch, $W_{-1}(x)$ is defined in the interval $[-e^{-1}, 1]$ and has a negative singularity for $x \rightarrow 0^{-1}$. The upper branch, $W_0(x)$, is defined for $x \in [-e^{-1}, \infty]$. Hence, equation (28) can be written as:

$$W\left(- (1 + \alpha)(1 - u)e^{-(\alpha + 1)}\right) = -\frac{\alpha + e^{-\frac{\theta}{\lambda}(e^{\lambda x} - 1)}}{e^{-\frac{\theta}{\lambda}(e^{\lambda x} - 1)}} \tag{29}$$

Now for any $\alpha > 0$ and $u \in (0, 1)$, it follows that $\frac{\alpha + e^{-\frac{\theta}{\lambda}(e^{\lambda x} - 1)}}{e^{-\frac{\theta}{\lambda}(e^{\lambda x} - 1)}} > 1$ and $\left((1 + \alpha)(1 - u)e^{-(\alpha + 1)} \right) < 0$.

Therefore, considering the lower branch of the Lambert function, equation (29) can be presented as:

$$W_{-1}\left(- (\alpha + 1)(1 - u)e^{-(\alpha + 1)}\right) = -\frac{\alpha + e^{-\frac{\theta}{\lambda}(e^{\lambda x} - 1)}}{e^{-\frac{\theta}{\lambda}(e^{\lambda x} - 1)}} \tag{30}$$

Collecting like terms in equation (30) and simplifying the result, the quantile function of the OLinGomD is obtained as:

$$Q(u) = \frac{1}{\lambda} \log \left\{ 1 + \frac{\lambda}{\theta} \log \left[-\frac{1}{\alpha} - \frac{1}{\alpha} W_{-1}\left(- (1 + \alpha)(1 - u)e^{-(\alpha + 1)}\right) \right] \right\} \tag{31}$$

where u is a uniform variate on the unit interval $(0, 1)$ and $W_{-1}(\cdot)$ represents the negative branch of the Lambert function.

The median of X from the OLinGomD is simply obtained by setting $u=0.5$ and this substitution of $u = 0.5$ in equation (31) gives:

$$Median = \frac{1}{\lambda} \log \left\{ 1 + \frac{\lambda}{\theta} \log \left[-\frac{1}{\alpha} - \frac{1}{\alpha} W_{-1} \left(-(1+\alpha) \left(\frac{1}{2}\right) e^{-(\alpha+1)} \right) \right] \right\} \quad (32)$$

Also, random numbers can be simulated from the OLinGomD by setting $Q(u) = X$ and this process is called simulation using inverse transformation method. This means for any $\theta, \lambda, \alpha > 0$ and $u \in (0, 1)$:

$$X = \frac{1}{\lambda} \log \left\{ 1 + \frac{\lambda}{\theta} \log \left[-\frac{1}{\alpha} - \frac{1}{\alpha} W_{-1} \left(-(1+\alpha)(1-u) e^{-(\alpha+1)} \right) \right] \right\} \quad (33)$$

“where u is a uniform variate on the unit interval $(0, 1)$ and $W_{-1}(\cdot)$ represents the negative branch of the Lambert function”.

According to [24], the Bowley’s measure of skewness based on quartiles is defined as:

$$SK = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{2}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \quad (34)$$

And [25] presented the Moors’ kurtosis based on octiles by:

$$KT = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) - Q\left(\frac{3}{8}\right) + \left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{1}{8}\right)} \quad (35)$$

“where $Q(\cdot)$ is calculated by using the quantile function from equation (31).

3.3 Distribution of order statistics

Suppose X_1, X_2, \dots, X_n is a random sample from the OLinGomD and let $X_{1:n}, X_{2:n}, \dots, X_{i:n}$ denote the corresponding order statistic obtained from this same sample. The pdf, $f_{i:n}(x)$ of the i^{th} order statistic can be obtained by:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} f(x) F(x)^{k+i-1} \quad (36)$$

Using (5) and (6), the pdf of the i^{th} order statistics $X_{i:n}$, can be expressed from (36) as:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} \left[\frac{\alpha^2 \theta e^{\lambda x} e^{\frac{\theta}{\lambda}(e^{\lambda x-1})}}{(1+\alpha)} e^{\left[1 - \frac{\theta}{\lambda}(e^{\lambda x-1})\right]} \right] \left[1 - \frac{\left(\alpha + e^{-\frac{\theta}{\lambda}(e^{\lambda x-1})}\right) e^{\left[1 - \frac{\theta}{\lambda}(e^{\lambda x-1})\right]}}{(1+\alpha) e^{-\frac{\theta}{\lambda}(e^{\lambda x-1})}} \right]^{i+k-1} \quad (37)$$

Hence, the pdf of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the OLinGomD are respectively given by:

$$f_{ln}(x) = n \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \left[\frac{\alpha^2 \theta e^{\lambda x} e^{2\frac{\theta}{\lambda}(e^{\lambda x}-1)}}{(1+\alpha)} e^{\alpha \left[1 - e^{\frac{\theta}{\lambda}(e^{\lambda x}-1)} \right]} \right] \left[1 - \frac{(\alpha + e^{-\frac{\theta}{\lambda}(e^{\lambda x}-1)}) e^{\alpha \left[1 - e^{\frac{\theta}{\lambda}(e^{\lambda x}-1)} \right]}}{(1+\alpha) e^{\frac{\theta}{\lambda}(e^{\lambda x}-1)}} \right]^k \quad (38)$$

and

$$f_{nn}(x) = n \left[\frac{\alpha^2 \theta e^{\lambda x} e^{2\frac{\theta}{\lambda}(e^{\lambda x}-1)}}{(1+\alpha)} e^{\alpha \left[1 - e^{\frac{\theta}{\lambda}(e^{\lambda x}-1)} \right]} \right] \left[1 - \frac{(\alpha + e^{-\frac{\theta}{\lambda}(e^{\lambda x}-1)}) e^{\alpha \left[1 - e^{\frac{\theta}{\lambda}(e^{\lambda x}-1)} \right]}}{(1+\alpha) e^{\frac{\theta}{\lambda}(e^{\lambda x}-1)}} \right]^{n-1} \quad (39)$$

4 Estimation of Unknown Parameters of the OLinGomD

Let X_1, X_2, \dots, X_n be a sample of size "n" independently and identically distributed random variables from the OLinGomD with unknown parameters, α , θ and λ defined previously.

The likelihood function is given by:

$$L(\underline{X} | \alpha, \theta, \lambda) = \frac{(\alpha^2 \theta)^n e^{\lambda \sum_{i=1}^n x_i} e^{2\frac{\theta}{\lambda} \sum_{i=1}^n (e^{\lambda x_i}-1)}}{(1+\alpha)^n} e^{\alpha \sum_{i=1}^n \left[1 - e^{\frac{\theta}{\lambda}(e^{\lambda x_i}-1)} \right]}$$

Let the log-likelihood function be $l = \log L(\underline{X} | \alpha, \theta)$ therefore

$$l = 2n \log \alpha + n \log \theta - n \log(1+\alpha) + \lambda \sum_{i=1}^n x_i + \frac{2\theta}{\lambda} \sum_{i=1}^n (e^{\lambda x_i} - 1) + \alpha \sum_{i=1}^n \left[1 - e^{\frac{\theta}{\lambda}(e^{\lambda x_i}-1)} \right] \quad (40)$$

Differentiating l partially with respect to α , θ and λ respectively gives;

$$\frac{\partial l}{\partial \alpha} = \frac{2n}{\alpha} - \frac{n}{(1+\alpha)} + \sum_{i=1}^n \left[1 - e^{\frac{\theta}{\lambda}(e^{\lambda x_i}-1)} \right] \quad (41)$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} + \frac{2}{\lambda} \sum_{i=1}^n (e^{\lambda x_i} - 1) - \frac{\alpha}{\lambda} \sum_{i=1}^n \left[(e^{\lambda x_i} - 1) e^{\frac{\theta}{\lambda}(e^{\lambda x_i}-1)} \right] \quad (42)$$

$$\frac{\partial l}{\partial \lambda} = \sum_{i=1}^n x_i + \frac{2\theta}{\lambda} \sum_{i=1}^n (x_i e^{\lambda x_i} - \lambda^{-1} (e^{\lambda x_i} - 1)) - \frac{\alpha \theta}{\lambda} \sum_{i=1}^n \left[(x_i e^{\lambda x_i} - \lambda^{-1} (e^{\lambda x_i} - 1)) e^{\frac{\theta}{\lambda}(e^{\lambda x_i}-1)} \right] \quad (43)$$

Equating (41), (42) and (43) to zero (0) and solving for the solution of the non-linear system of equations will give the maximum likelihood estimates (MLEs) of parameters α , θ and λ .

5 Applications

This section presents two datasets, their descriptive statistics and applications of some selected extensions of the Rayleigh distribution. The section compares the performance of the Odd Lindley-Gompertz distribution (OLinGomD) to that of Generalized Gompertz distribution (GenGomD), Lomax-Gompertz distribution (LomGomD), Transmuted Gompertz distribution (TGomD), Power Gompertz distribution (PGomD) and the classical Gompertz distribution (GomD).

Aiming to evaluate the performance of the models listed above, the Akaike Information Criterion (AIC) is being used in this article as previously considered by [15]. The formula for this statistic is given as:

$$AIC = -2ll + 2k$$

Where ll denotes the value of the log-likelihood evaluated at the maximum likelihood estimates (MLEs) and k is the number of model parameters.

Meanwhile the model with the smallest value of AIC is to be considered as the best model that fit the data.

Dataset I: This dataset represents the lifetime's data relating to relief times (in minutes) of 20 patients receiving an analgesic and reported by [26] and has also been used by [27,28,29,30] and [31].

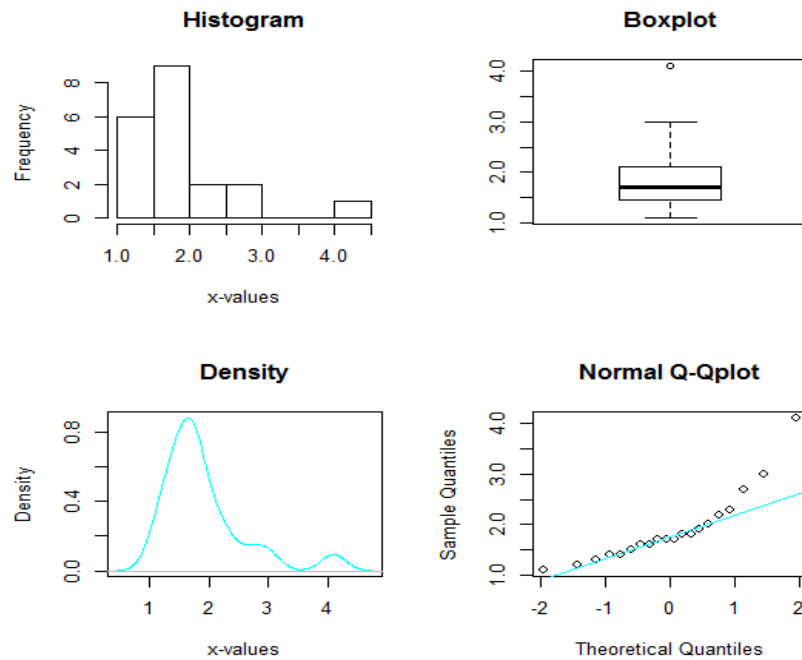


Fig. 2. A graphical summary dataset I

Table 1. Summary statistics for dataset I

Parameters	n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
Values	20	1.10	1.475	1.70	2.05	1.90	4.10	0.4958	1.8625	7.1854

Based on the descriptive statistics in Table 1 and Fig. 2, it is clear that the first dataset (dataset I) is skewed to the right or positively skewed and could be good for flexible models like OLinGomD.

Table 2. Performance of the fitted distributions based on the value of AIC using dataset I

Distributions	Parameter estimates	Log-likelihood value	AIC	Rank of models
OLinGomD	$\hat{\theta} = 1.1183$ $\hat{\lambda} = 0.1854$ $\hat{\alpha} = 0.1676$	161.8985	-317.7970	1 st
GenGomD	$\hat{\theta} = 0.0797$ $\hat{\lambda} = 1.7179569$ $\hat{\alpha} = 1.4005451$	115.988	-225.9759	2 nd
LomGomD	$\hat{\theta} = 0.04256$ $\hat{\lambda} = 2.133$ $\hat{\alpha} = 3.561$ $\hat{\beta} = 3.323$	105.7131	-203.4261	3 rd
TGomD	$\hat{\theta} = 0.5493$ $\hat{\lambda} = 0.4126$ $\hat{\alpha} = -0.9593$	24.11346	-42.22693	4 th
PGomD	$\hat{\theta} = 0.11856$ $\hat{\lambda} = 0.02237$ $\hat{\alpha} = 2.15954$	-19.12861	44.25723	5 th
GomD	$\hat{\theta} = 0.14509$ $\hat{\lambda} = 0.69852$	-24.5907	53.18141	6 th

Table 2 presents the parameter estimates and the values of AIC for the six fitted models using dataset I. The values in the above table reveal that the OLinGomD has a smaller value of AIC compared to the other five distributions and is therefore taken to fit the dataset better than the other models.

From the estimated pdf and cdf plots in Fig. 3, it is very clear that the OLinGomD fits the data much better than the other five distributions. This performance shows that the OLinGomD is flexible and can take different shapes or fit various datasets.

Looking at the probability plots in Fig. 4, it can be seen that the data points are closer to the straight line of the plot for OLinGomD than the other distributions which is a confirmation that the OLinGomD has a better fit for the data as already demonstrated in Table 2 and Fig. 3.

Data set II: This dataset represents the strength of 1.5cm glass fibers initially collected by members of staff at the UK national laboratory. It has been used by [32,33,34,35,36,37] as well as [38]. Its summary is given as follows:

Table 3. Descriptive statistics for dataset II

n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
63	0.550	1.375	1.590	1.685	1.507	2.240	0.105	-0.8786	3.9238

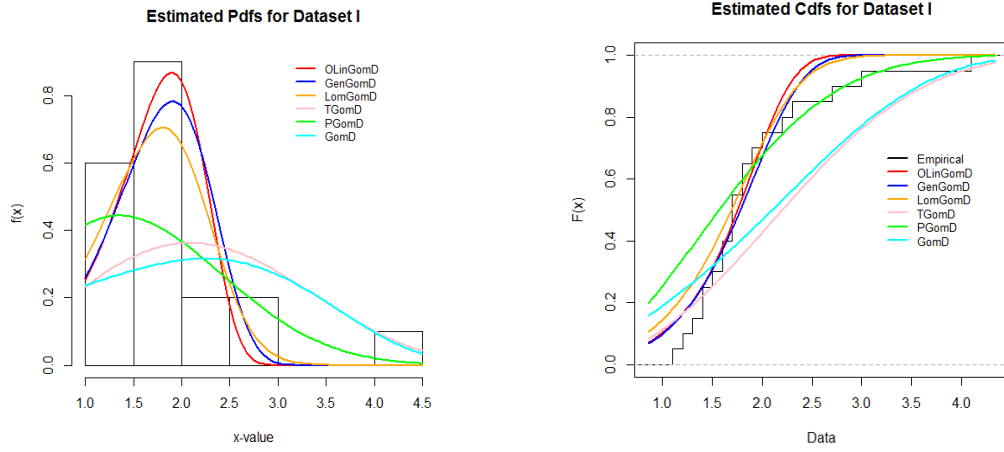


Fig. 3. Histogram and Plots of the Estimated Densities and Cdfs of the Fitted Distributions based on Datasets I

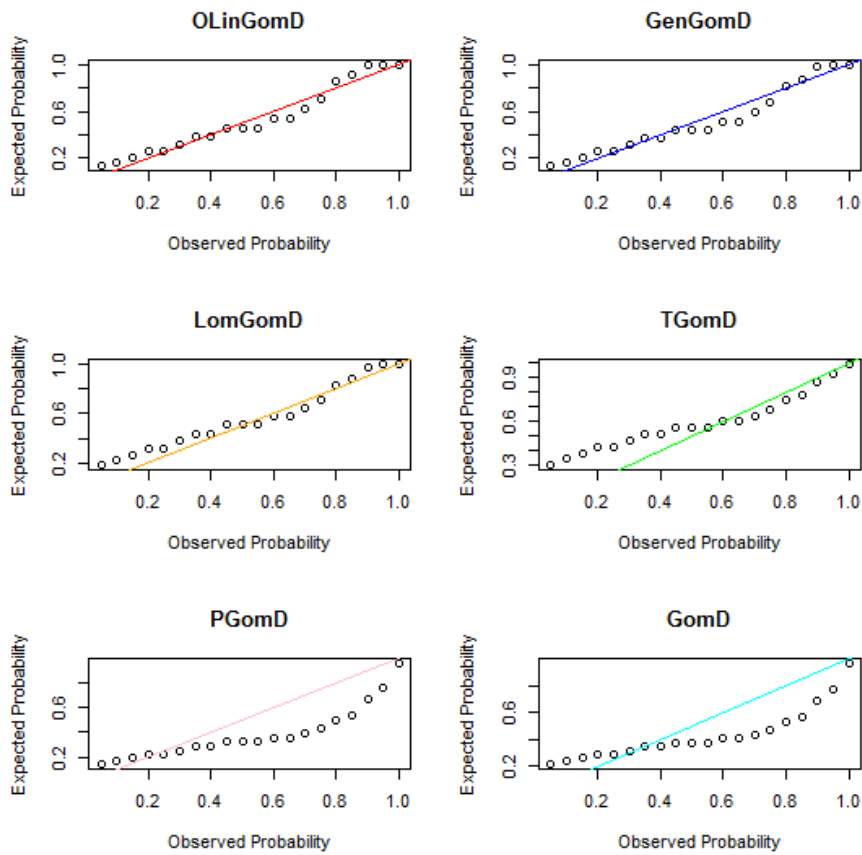


Fig. 4. Probability plots for the fitted distributions based on dataset I

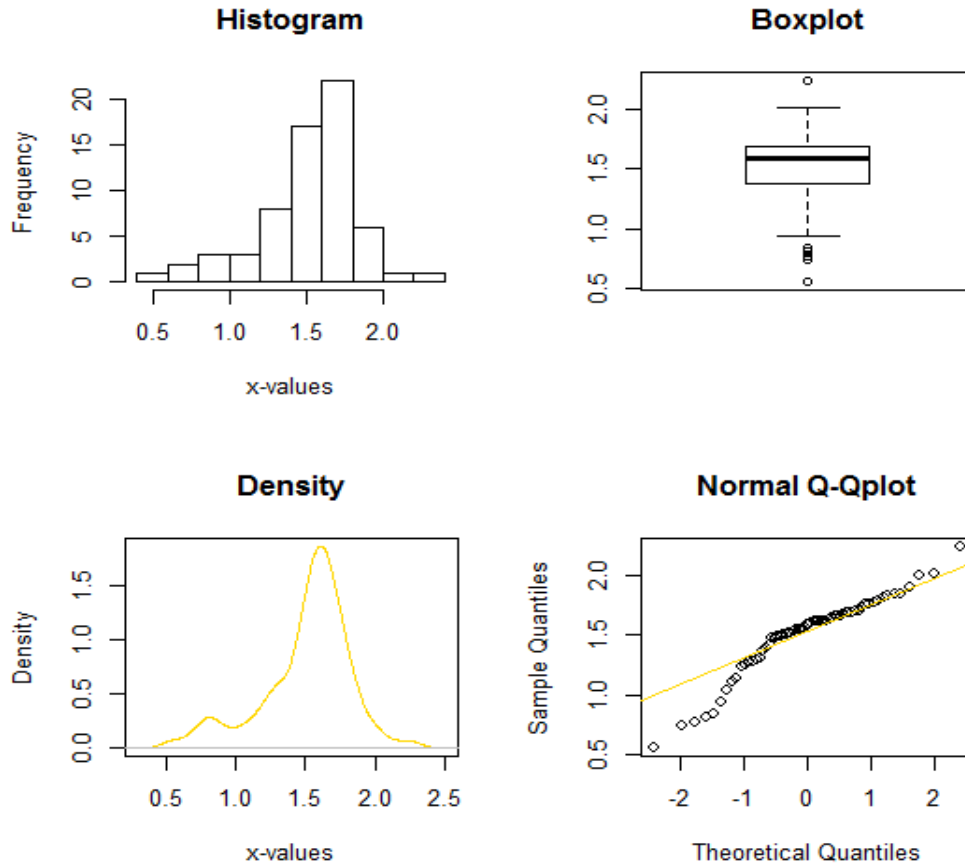


Fig. 5. A graphical summary dataset II

Looking at the descriptive statistics in Table 3 and the histogram, box plot, density and normal Q-Q plot shown in Fig. 5 above, it is revealed that second dataset (dataset II) is negatively skewed, that is, skewed to the left and could also be suitable for skewed distributions like the OLinGomD.

Also the result in Table 4 presents the parameter estimates and the values of AIC for the six fitted models using dataset II. From this table, it is noticed that the OLinGomD has a smaller value of AIC compared to the other five distributions and is therefore taken to fit the second dataset better than the other five fitted models.

Also, looking at the estimated pdf and cdf plots in Fig. 6, the plots indicate that the OLinGomD fits the second data (dataset II) better than the other five distributions. This performance shows that the OLinGomD is flexible and can take different shapes or fit various datasets.

Considering the probability plots in Fig. 7, it can be noted that the points in the p-p plot are closer to the straight line of the plot for OLinGomD than the other distributions which is a confirmation that the OLinGomD has a better fit for the data as already demonstrated in Table 4 and Fig. 6.

Table 4. Performance of the fitted distributions based on the value of AIC using dataset II

Distributions	Parameter estimates	Log-likelihood value	AIC	Rank of models
OLinGomD	$\hat{\theta} = 0.7452$ $\hat{\lambda} = 2.0747$ $\hat{\alpha} = -0.9999$	535.9007	-1065.8013	1 st
LomGomD	$\hat{\theta} = 0.0002651$ $\hat{\lambda} = 5.4073836$ $\hat{\alpha} = 3.2158189$ $\hat{\beta} = 6.1656918$	452.9747	-897.9494	2 nd
GenGomD	$\hat{\theta} = 0.001143$ $\hat{\lambda} = 5.447001$ $\hat{\alpha} = 3.531687$	299.0638	-592.1276	3 rd
TGomD	$\hat{\theta} = 1.07580$ $\hat{\lambda} = 0.43643$ $\hat{\alpha} = -0.090758$	88.33262	-170.6652	4 th
GomD	$\hat{\theta} = 0.008418$ $\hat{\lambda} = 3.689379$	-14.57588	33.15175	5 th
PGomD	$\hat{\theta} = 0.02154$ $\hat{\lambda} = 2.46556$ $\hat{\alpha} = 1.30117$	-15.27265	36.54529	6 th

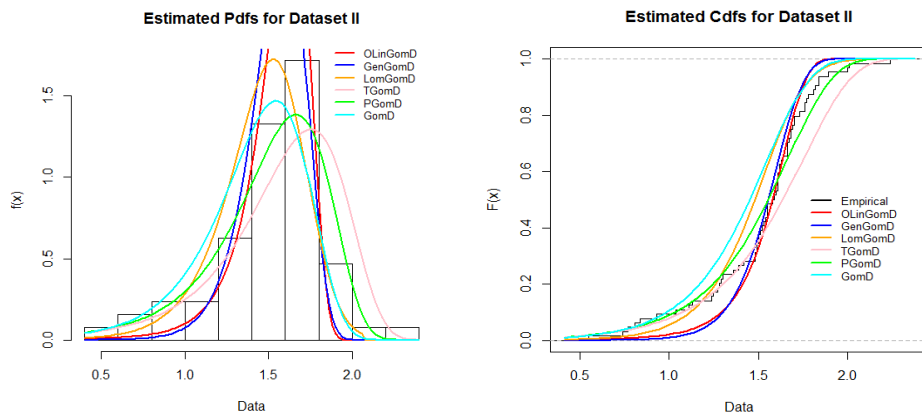


Fig. 6. Histogram and plots of the estimated densities and Cdfs of the fitted distributions based on datasets II

With reference to the results in Table 2 and Table 4 above and comparing the values of the *AIC* for the six fitted distributions, it is obvious that the OLinGomD fits the two datasets better than the other five distributions. This is based on the decision statement which says that the distribution with a smaller value of the test statistic (*AIC*) will be considered as the most efficient model as considered in this study.

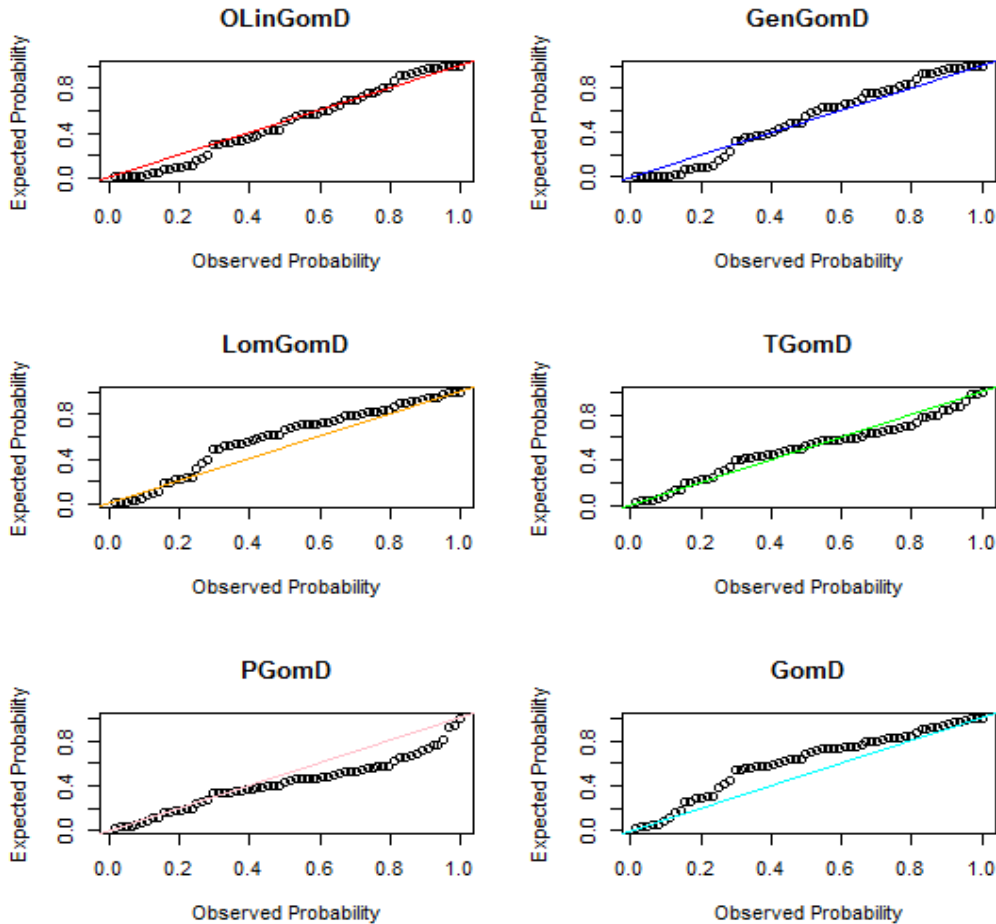


Fig. 7. Probability plots for the fitted distributions based on dataset II

Also, the estimated pdfs and cdfs displayed in Fig. 3 for dataset I and Fig. 6 for dataset II as well as the P-P plots presented in Fig. 4 for dataset I and Fig. 7 for dataset II clearly support and confirm the results in Table 2 and Table 4.

Relating the present result to the result of [22], it can be seen that the odd Lindley-G family by [14] is a very good method for defining compound distributions with greater level of performance and flexibility irrespective of the nature of the datasets to be analyzed.

6 Summary and Conclusion

In this paper, the three-parameter model named the “odd Lindley-Gompertz distribution” introduced previously by [22] has been studied and evaluated extensively. The article has provided a comprehensive

study of some mathematical and statistical properties of the odd Lindley-Gompertz distribution which include the derivation of explicit expressions for its ordinary moment, moment generating function, characteristics function, cumulant generating function, survival function, hazard function, reverse hazard rate, cumulative hazard function, odds function and the quantile function which is useful for obtaining the median, skewness and kurtosis and simulation of random numbers from the OLinGomD. It also obtained the density function of its minimum and maximum order statistics. The estimation of the unknown model parameters using the method of maximum likelihood estimation has also been considered. The performance of the odd Lindley-Gompertz distribution has also been checked in this present article using two real life datasets and the results reveal that the odd Lindley-Gompertz distribution is more flexible compared to the other fitted distributions due to its ability to fit the two datasets better than the Generalized Gompertz distribution (GenGomD), Lomax-Gompertz distribution (LomGomD), Transmuted Gompertz distribution (TGomD), Power Gompertz distribution (PGomD) and the classical Gompertz distribution (GomD).

Competing Interests

Authors have declared that no competing interests exist.

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