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A Study of Second Order Slope Rotatable Designs under Intra-class Correlated Structure of Errors Using Partially Balanced Incomplete Block Type Designs

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Authors' contributions

This work was carried out in collaboration between both authors. Author BS designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author BRV managed the analyses of the study and managed the literature searches. Both authors read and approved the final manuscript.

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Abstract

In this paper, a study of second order slope rotatable designs under intra-class correlated structure of errors using partially balanced incomplete block type designs is suggested. Further, we study the variance function of the estimated slopes for different values of intra-class correlated coefficient (ρ) and distance from centre (d) for $6 \le v \le 12$ (number of factors) are also suggested. We note the new method sometimes leads to designs with fewer number of design points.

Keywords: Second order response surface designs; slope rotatability; intra-class correlated errors.

1 Introduction

Response surface methodology (RSM) often deals with a natural and desirable property rotatability, which requires that, the variance of the predicted response at a point remains constant at all such points that are

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equidistant from the design centre. To achieve stability in prediction variance, this important property of rotatability was developed. Analogous to rotatability, the concept of slope-rotatability has been progressed. The idea of slope-rotatability is an important design criterion for response surface design. Recently, in the design of experiments for response surface analysis, attention has been focused on the estimation of differences in response rather than absolute value of the response mean itself. The slope-rotatable design is that of which the variance of partial derivative is only a function of the design centre.

In the context of response surface methodology, Box and Hunter [1] introduced the concept of rotatability assuming errors are uncorrelated and homoscedastic. Das and Narasimham [2] constructed rotatable designs through balanced incomplete block designs (BIBD). Chowdhury and Gupta [3] constructed second order rotatable designs (SORD) associated with partially balanced incomplete block designs. Hader and Park [4] constructed slope rotatable central composite designs. Victorbabu and Narasimham [5,6,7] constructed second order slope rotatable designs (SOSRD) through BIBD, a pair of incomplete block designs and partially balanced incomplete block (PBIB) type designs respectively. Victorbabu [8] suggested a review on SOSRD.

So far all the authors studied rotatable and slope rotatable designs assuming errors to be uncorrelated and homoscedastic. However, it is not uncommon to come across the practical situations when errors are correlated, violating usual assumptions. Panda and Das [9] studied first order rotatable designs with correlated errors. Das [10,11] introduced and studied robust second order rotatable designs (RSORD). Das [12,13,14] studied slope rotatability with correlated errors. Rajyalakshmi [15] suggested some contributions to second order rotatable and slope rotatable designs under different correlated error structures. Rajyalakshmi and Victorbabu [16,17,18] constructed SOSRD under intra-class correlation structure of errors using central composite designs (CCD), symmetrical unequal block arrangements (SUBA) with two unequal block sizes and BIBD respectively. Rajyalakshmi and Victorbabu [19,20] constructed SOSRD under intra-class correlation structure of errors using SUBA with two unequal block sizes and BIBD respectively. Sulochana and Victorbabu [21,22] studied SOSRD under intra-class correlated structure of errors using a pair of BIBD and a pair of SUBA with two unequal block sizes respectively. Sulochana and Victorbabu studied SOSRD under tri-diagonal correlated structure of errors using a pair of SUBA with two unequal block sizes.

Partially balanced incomplete block designs: Partially balanced incomplete block designs play important role in design of experiments, especially in field of experiments where balanced incomplete block design required a large number of blocks.

An arrangement of v treatments in a blocks of size k each (k<v) is called partially balanced incomplete block design (PBIBD) [23], if

1. Each treatment has the same numbers of replications.
2. Some pairs of treatments have λ_1 replications some pair of treatment have λ_2 replications. Some pair of treatments has λ_3 replications.
3. Two treatments which occur together λ_i times are called i^{th} associates. Then the number of i^{ih} for any treatments θ is same (n_i).
4. If θ, ϕ are k^{th} associates (k = 1, 2,…, n) and if $p_{ij(k)}$ is the number of treatments common to i^{th} associates of θ and j^{th} associates of ϕ (i = 1,2,…m) (j = 1,2,…,n) then the p matrix ($p_{ij(k)}$) must same for each θ and ϕ.

The above PBIBD is called m-associate PBIBD in particular m = 2 then it is called two associate PBIBD.

In this paper following the works of Das [12], Rajyalakshmi and Victorbabu [16,17,18] here a study of SOSRD under intra-class correlated structure of errors using PBIB type designs is suggested. Further we study the variance function of the estimated slopes for different values of intra-class correlated coefficient (ρ) and also obtain the distance from centre (d) for $6 \leq v \leq 12$ (v number of factors).

2 Conditions for Second Order Slope Rotatable Designs under Intra-class Correlated Structure of Errors (cf. Das [12], Rajyalakshmi and Victorbabu [16,17,18])

A second order response surface design $D=((X_{iu}))$ for fitting,

$$Y_u(x)=b_0+\sum_{i=1}^{v} b_i X_{iu}+\sum_{i=1}^{v} b_{ii} X_{iu}^2+\sum_{i=1}^{v}\sum_{i<j}^{v} b_{ij} X_{iu} X_{ju}+e_u \tag{2.1}$$

where X_{iu} denotes the level of the ith factor (i=1,2,…,v) in the uth run (u=1,2,…,N) of the experiment, e_u's are correlated random errors, is said to be a SOSRD under intra-class correlated structure of errors, if the variance of the estimate of first order partial derivative of $Y_u(X_{1u},X_{2u},X_{3u},...,X_{vu})$ with respect to each independent variable X_i is only a function of the distance $\left(d^2=\sum_{i=1}^{v} X_i^2\right)$ of the point $(X_{1u},X_{2u},X_{3u},...,X_{vu})$ from the origin (centre of the design). Such a spherical variance function for estimation of slopes in the second order response surface is achieved if the design points satisfy the following conditions (cf. Das [10,11,12], Rajyalakshmi and Victorbabu [16,17,19]).

The necessary and sufficient conditions for second order slope rotatability under intra-class correlated structure of errors are:

$$\sum_{u=1}^{N} X_{iu}=0, \sum_{u=1}^{N} X_{iu}X_{ju}=0, \sum_{u=1}^{N} X_{iu}X_{ju}^2=0, \sum_{u=1}^{N} X_{iu}X_{ju}X_{ku}=0, \sum_{u=1}^{N} X_{iu}^3=0,$$

$$\sum_{u=1}^{N} X_{iu}X_{ju}^3=0, \sum_{u=1}^{N} X_{iu}X_{ju}X_{ku}^2=0, \sum_{u=1}^{N} X_{iu}X_{ju}X_{ku}X_{lu}=0, \text{ for } i\neq j\neq k\neq l; \tag{2.2}$$

$$\sum_{u=1}^{N} X_{iu}^2=\text{constant}=N\mu_2 \tag{2.3}$$

$$\sum_{u=1}^{N} X_{iu}^4=\text{constant}= cN\mu_4, \text{ for all } i \tag{2.4}$$

$$\sum_{u=1}^{N} X_{iu}^2X_{ju}^2=\text{constant}=N\mu_4, \text{ for all values } i\neq j \tag{2.5}$$

From (2.4) and (2.5), we have,

$$\sum_{u=1}^{N} X_{iu}^4= c\sum_{u=1}^{N} X_{iu}^2X_{ju}^2 \tag{2.6}$$

where c, μ_2 and μ_4 are constants. The summation is over the designs points.

Using the above simple symmetric conditions, the variances and covariances of the estimated parameters under the intra-class correlated structure of errors are as follows:

$$V\left(\hat{b}_0\right)=\frac{[\mu_4\left(c+v-1\right)A-v\rho N\mu_2^2]\sigma^2 A}{N\Delta} \tag{2.7}$$

$$V\left(\hat{b}_i\right)=\frac{\sigma^2(1-\rho)}{N\mu_2} \tag{2.8}$$

$$V\left(\hat{b}_{ij}\right)=\frac{\sigma^2(1-\rho)}{N\mu_4} \tag{2.9}$$

$$V\left(\hat{b}_{ii}\right)=\frac{\sigma^2(1-\rho)[\mu_4\left(c+v-2\right)A-(v-1)\rho N\mu_2^2-(v-1)\mu_2^2(1-\rho)]}{\left(c-1\right)N\mu_4\Delta} \tag{2.10}$$

$$Cov\left(\hat{b}_0,\hat{b}_{ii}\right)=\frac{-\sigma^2\mu_2^2(1-\rho)A}{N\Delta} \tag{2.11}$$

$$Cov\left(\hat{b}_{ii},\hat{b}_{ij}\right)=\frac{\sigma^2(1-\rho)[\mu_2^2\left(1-\rho\right)-\mu_4 A+\rho N\mu_2^2]}{\left(c-1\right)N\mu_4\Delta} \tag{2.12}$$

where $A=\left\{1+\left(N-1\right)\rho\right\}$, $\Delta=\left[\mu_4\left(c+v-1\right)A-v\rho N\mu_2^2-v\mu_2^2\left(1-\rho\right)\right]$

and the other covariances are zero.

An inspection of the variance of $\hat{b}_0$ shows that a necessary condition for the existence of a non-singular second order slope rotatable design under intra-class correlated structure of errors is

$$\mu_4\left(c+v-1\right)A-v\rho N\mu_2^2-v\mu_2^2\left(1-\rho\right)>0 \tag{2.13}$$

From (2.13), we have,

$$\frac{\mu_4}{\mu_2^2}>\frac{v}{c+v-1} \text{ (non-singularity condition)} \tag{2.14}$$

If the non-singularity condition (2.14) exists then only the design exists.

For the second order model,

$$\frac{\partial \hat{Y}}{\partial x_i} = \hat{b}_i + 2\hat{b}_{ii} X_i + \sum_{i=1, j\neq i}^{v} \hat{b}_{ij} X_j,$$

$$V\left(\frac{\partial \hat{Y}}{\partial x_i}\right) = V\left(\hat{b}_i\right) + 4X_i^2 V\left(\hat{b}_{ii}\right) + \sum_{i=1, j\neq i}^{v} X_j^2 V\left(\hat{b}_{ij}\right) \tag{2.15}$$

The condition for right hand side of equation (2.15) to be a function of $(d^2 = \sum_{i=1}^{v} X_i^2)$ alone (for slope rotatability) is clearly,

$$V\left(\hat{b}_{ii}\right) = \frac{1}{4} V\left(\hat{b}_{ij}\right) \tag{2.16}$$

On simplification of (2.16) using (2.9) and (2.10) leads to

$$\left(\frac{AcN\mu_4 - B}{1-\rho}\right)\left[4N - \left(\frac{AcN\mu_4 - B}{A\mu_4}\right) v \left(\frac{N\mu_2^2(1-\rho)}{A\mu_4}\right) - (v-2)\left(\frac{AN\mu_4 - B}{A\mu_4}\right)\right] +$$
$$\left(\frac{AN\mu_4 - B}{1-\rho}\right)\left[4(v-2) - (v-1)\left(\frac{AN\mu_4 - B}{AN\mu_4}\right)\right] - N^2\mu_2^2\left[4(v-1) + v\left(\frac{AN\mu_4 - B}{AN\mu_4}\right)\right] = 0 \tag{2.17}$$

$A = \{1 + (N-1)\rho\}$, $B = \rho N^2 \mu_2^2$

If $\rho = 0$ condition (2.17) is equal to

$$\mu_4\left[v(5-c) - (c-3)^2\right] + \mu_2^2\left[v(c-5) + 4\right] = 0 \tag{2.18}$$

which is similar to the SOSRD condition of Victorbabu and Narasimham [5].

Therefore, equations (2.2) to (2.12), (2.14) to (2.17) give a set of conditions for SOSRD under intra-class correlated structure of errors for any general second order response surface design. Further,

$$V\left(\frac{\partial \hat{y}_u}{\partial X_i}\right) = \frac{1}{N}\left(\frac{1}{\mu_2} + \frac{d^2}{\mu_4}\right)(1-\rho)\sigma^2 \tag{2.19}$$

3 Construction of SOSRD under Intra-class Correlated Structure of Errors Using Partially Balanced Incomplete Block Type Designs

Following the methods of constructions, of Das [10,11,12], Victorbabu and Narasimham [5,6,7], Rajyalakshmi and Victorbabu [16,17,18]. The procedure for the construction SOSRD under intra-class correlated structure of errors using PBIB type designs involves ρ be the correlation errors of any two observations, each having the same σ^2.

Take an incomplete arrangement with constant block size and replication in which some pair of treatments occur λ_1 times each ($\lambda_1 \neq 0$) and some other pairs do not occur at all ($\lambda_2 = 0$) (the design need not be PBIBD). Take this as the first design. For the second design take the incomplete block design with all missing pairs (in the first design) once each with $k = 2,\ \lambda_1^{'}=0, \text{ and } \lambda_2^{'} = 1$. Such pairs of designs can be constructed in a straight forward manner using existing two- associate PBIB designs with one of the λ's equal to zero. The method of construction is suggested below.

Let $D_1=\left(v,b_1,r_1,k_1,\lambda_1,\lambda_2=0\right)$ be an incomplete block design with constant replication in which only some pair of treatments occur a constant number of times λ_1 ($\lambda_2 = 0$). Let $\left[1\text{-}\left(v,b_1,r_1,k_1,\lambda_1,\lambda_2=0\right)\right]$ denote the design points generated from the transpose of the incidence matrix of incomplete block design. $\left[1\text{-}\left(v,b_1,r_1,k_1,\lambda_1,\lambda_2=0\right)\right]2^{t(k_1)}$ are the $b_1 2^{t(k_1)}$ design points from D_1 by "multiplication" (see, Das and Narasimham [2]).

Let $D_2=\left(v,b_2,r_2,k_2,\lambda_1^{'}=0,\lambda_2^{'}=1\right)$ be the associated second design containing only the missing pairs of treatments of above D_1. $\left[a_1\text{-}\left(v,b_2,r_2,k_2,\lambda_1^{'}=0,\lambda_2^{'}=1\right)\right]2^2$ are the $b_2 2^2$ design points generated from D_2 by "multiplication". $\left(a,0,\ldots,0\right)2^1$ denote the design points generated from (a, 0,…,0) point set. n_0 be number of central points. The method of construction of SOSRD under intra-class correlated structure of errors using PBIB type designs is given in the following theorem.

Theorem: The design points,

$\left[1\text{-}\left(v,b_1,r_1,k_1,\lambda_1,\lambda_2=0\right)\right]2^{t(k_1)} \cup \left[a_1\text{-}\left(v,b_2,r_2,k_2,\lambda_1^{'}=0,\lambda_2^{'}=1\right)\right]2^2 \cup \left(a,0,\ldots,0\right)2^1 \cup n_0$ give a v-dimensional SOSRD under intra-class correlated structure of errors in $N=b_1 2^{t(k_1)}+b_2 2^2+2v+n_0$ design points. Where a^2 is a positive real root of the fourth degree polynomial equation,

$$
\begin{aligned}
&\left[8v-4N\right]A^2a^8+8v\left[r_1 2^{t(k_1)}+4r_2\sqrt{\lambda_1 2^{t(k_1)-2}}\right]A^2a^6\\
&\left[\begin{array}{l}2^{t(k_1)+1}N\left(\lambda_1\left(6-v\right)-2\left(r_1+r_2\lambda_1\right)\right)+\\ 2^{t(k_1)+1}\left[\left(v\left(r_1^2 2^{t(k_1)}+2r_2\lambda_1\right)+2\left(r_1+2r_2^2\lambda_1-5\lambda_1\right)+8r_1r_2\sqrt{\lambda_1 2^{t(k_1)-2}}\right)+8\lambda_1\right]\end{array}\right]A^2a^4+\\
&\left[2^{2t(k_1)+2}r_1\left(v\left(r_1+r_2\lambda_1-5\lambda_1\right)+4\lambda_1\right)+2^{t(k_1)+4}r_2\left(v\left(r_1+r_2\lambda_1-5\lambda_1\right)+4\lambda_1\right)\right]A^2a^2+\\
&\left[\begin{array}{l}2^{2t(k_1)}N\left[v\lambda_1\left(5\lambda_1-r_1-r_2\lambda_1\right)-\lambda_1\left(r_2^2-6r_2+9\right)-r_1\left(r_1-6\lambda_1+2r_2\lambda_1\right)\right]+\\ \left[\left[2^{3t(k_1)}r_1^2+\left(2^{2(k_1)+2}r_2^2\lambda_1\right)+\left(2^{2(k_1)+3}r_1r_2\sqrt{\lambda_1 2^{t(k_1)-2}}\right)\right]\left(\left(v\left(r_1+r_2\lambda_1-5\lambda_1\right)+4\lambda_1\right)\right)\right.\end{array}\right]A^2=0
\end{aligned}
$$

where $A\{1+(N-1)\rho\}$ (3.1)

Proof: For the design points generated from designs D_1 and D_2 conditions (2.2) are true. Further, from (2.3) to (2.5) we have,

$$\sum_{u=1}^{N} X_{iu}^2 = r_1 2^{t(k_1)} + r_2 2^{k_2} a_1^2 + 2a^2 = N\mu_2 \quad (3.2)$$

$$\sum_{u=1}^{N} X_{iu}^4 = r_1 2^{t(k_1)} + r_2 2^{k_2} a_1^4 + 2a^4 = cN\mu_4 \quad (3.3)$$

$$\sum_{u=1}^{N} X_{iu}^2 X_{ju}^2 = \lambda_1 2^{t(k_1)} = \lambda_2^{'} 2^{k_2} a_1^4 = N\mu_4 \quad (3.4)$$

From (3.4) we get, $a_1^4 = \dfrac{\lambda_1}{\lambda_2^{'}} 2^{t(k_1)-k_2} = \lambda_1 2^{k_1-2}$ (since $\lambda_2^{'}=1,\ k_2=2$)

By substituting (3.2), (3.3) and (3.4) in (2.17) and on simplification, we get (3.1). The design exists only if at least one positive real root exists for equation (3.1). Solving (3.1) we get SOSRD with intra-class correlated structure of errors using PBIB type designs with different values of 'a' for the 'v' different factors. The variance of estimated slopes of these SOSRD under intra-class correlated structure of errors for $0 \le \rho \le 0.9$ and for $6 \le v \le 12$ factors are given in Table 1.

Table 1. The variances of estimated derivatives (slopes) for $6 \le v \le 12$ factors of SOSRD under intra-class correlated structure of errors using PBIBD

ρ	($v=6, b_1=4, r_1=2, k_1=3 \lambda_1=1, \lambda_2=0$) ($v=6, b_2=3, r_2=6, k_2=\lambda_1^{'}=0, \lambda_2^{'}=1$) N = 57, a = 2.2475	($v=8, b_1=8, r_1=3, k_1=3, \lambda_1=1, \lambda_2=0$) ($v=8, b_2=4, r_2=1, k_2=2, \lambda_1^{'}=0, \lambda_2^{'}=1$) N = 97, a = 2.1811	($v=9, b_1=9, r_1=3, k_1=3, \lambda_1=1, \lambda_2=0$) ($v=9, b_2=9, r_2=2, k_2=2, \lambda_1^{'}=0, \lambda_2^{'}=1$) N = 127, a = 1.9735
	$V\left(\frac{\partial \hat{y}_u}{\partial X_i}\right)$	$V\left(\frac{\partial \hat{y}_u}{\partial X_i}\right)$	$V\left(\frac{\partial \hat{y}_u}{\partial X_i}\right)$
0	$0.0315\sigma^2+0.125\ \sigma^2 d^2$	$0.0255\ \sigma^2+0.125\ \sigma^2 d^2$	$0.0232\ \sigma^2+0.125\ \sigma^2 d^2$
0.1	$0.0283\ \sigma^2+0.1125\ \sigma^2 d^2$	$0.023\ \sigma^2+0.1125\ \sigma^2 d^2$	$0.02088\sigma^2+0.1125\ \sigma^2 d^2$
0.2	$0.0252\ \sigma^2+0.1\ \sigma^2 d^2$	$0.0204\ \sigma^2+0.1\ \sigma^2 d^2$	$0.01856\ \sigma^2+0.1\ \sigma^2 d^2$
0.3	$0.022\ \sigma^2+0.0875\ \sigma^2 d^2$	$0.0179\ \sigma^2+0.0875\ \sigma^2 d^2$	$0.01624\ \sigma^2+0.0875\ \sigma^2 d^2$
0.4	$0.0189\ \sigma^2+0.075\ \sigma^2 d^2$	$0.0153\ \sigma^2+0.075\ \sigma^2 d^2$	$0.01392\ \sigma^2+0.075\ \sigma^2 d^2$
0.5	$0.0157\ \sigma^2+0.0625\ \sigma^2 d^2$	$0.0127\ \sigma^2+0.0625\ \sigma^2 d^2$	$0.0116\ \sigma^2+0.0625\ \sigma^2 d^2$
0.6	$0.0126\ \sigma^2+0.05\ \sigma^2 d^2$	$0.0102\ \sigma^2+0.05\ \sigma^2 d^2$	$0.00928\ \sigma^2+0.05\ \sigma^2 d^2$
0.7	$0.0094\ \sigma^2+0.0375\ \sigma^2 d^2$	$0.0077\ \sigma^2+0.0375\ \sigma^2 d^2$	$0.00696\ \sigma^2+0.0375\ \sigma^2 d^2$
0.8	$0.0063\ \sigma^2+0.0025\ \sigma^2 d^2$	$0.0051\ \sigma^2+0.025\ \sigma^2 d^2$	$0.0046\ \sigma^2+0.025\ \sigma^2 d^2$
0.9	$0.0032\ \sigma^2+0.0125\ \sigma^2 d^2$	$0.0026\ \sigma^2+0.0125\ \sigma^2 d^2$	$0.00232\ \sigma^2+0.0125\ \sigma^2 d^2$

ρ	(v=10, b_1=8, r_1=4, k_1=5, λ_1=2,λ_2=0) (v=10, b_2=5, r_2=1, k_2=2, λ_1'=0, λ_2'=1) N = 169, a = 2.9568	(v=12, b_1=8, r_1=4, k_1=6, λ_1=2, λ_2=0) (v=12, b_2=6, r_2=1, k_2=2, λ_1'=0, λ_2'=1) N = 305, a = 2.9163
	$V\left(\frac{\partial \hat{y}_u}{\partial X_i}\right)$	$V\left(\frac{\partial \hat{y}_u}{\partial X_i}\right)$
0	$0.0115\ \sigma^2+0.0313\ \sigma^2 d^2$	$0.0062\ \sigma^2+0.0156\ \sigma^2 d^2$
0.1	$0.0104\ \sigma^2+0.0282\ \sigma^2 d^2$	$0.0056\ \sigma^2+0.0141\ \sigma^2 d^2$
0.2	$0.0092\ \sigma^2+0.025\ \sigma^2 d^2$	$0.0049\ \sigma^2+0.0125\ \sigma^2 d^2$
0.3	$0.0081\ \sigma^2+0.0219\ \sigma^2 d^2$	$0.0043\ \sigma^2+0.0109\ \sigma^2 d^2$
0.4	$0.0069\ \sigma^2+0.0188\ \sigma^2 d^2$	$0.0037\ \sigma^2+0.0094\ \sigma^2 d^2$
0.5	$0.0058\ \sigma^2+0.0157\ \sigma^2 d^2$	$0.0031\ \sigma^2+0.0078\ \sigma^2 d^2$
0.6	$0.046\ \sigma^2+0.0125\ \sigma^2 d^2$	$0.0025\ \sigma^2+0.0063\ \sigma^2 d^2$
0.7	$0.0035\ \sigma^2+0.0094\ \sigma^2 d^2$	$0.0019\ \sigma^2+0.0047\ \sigma^2 d^2$
0.8	$0.0023\ \sigma^2+0.0063\ \sigma^2 d^2$	$0.0012\ \sigma^2+0.0031\ \sigma^2 d^2$
0.9	$0.0012\ \sigma^2+0.0031\ \sigma^2 d^2$	$0.0006\ \sigma^2+0.0016\ \sigma^2 d^2$

Example: We illustrate the construction of SOSRD under intra-class correlated structure of errors for v = 6 factors with the help of PBIB type designs with parameters

D_1=(v=6, b_1=4, r_1=2, k_1=3, λ_1=1, λ_2=0) and the associated second design D_2 with missing pairs in D_1. Here D_1={ (1, 2, 3), (1, 5, 6), (2, 4, 6), (3, 4, 5)}

D_2={ (3, 6), (2, 5), (1, 4)}

The design points,

$$\left[1\text{-}\left(v{=}6,b_1{=}4,r_1{=}2,k_1{=}3,\lambda_1{=}1,\lambda_2{=}0\right)\right]2^3 \cup$$
$$\left[a_1-\left(v{=}6,b_2{=}3,r_2{=}1,k_2{=}2,\lambda_1'{=}0,\lambda_2'{=}1\right)\right]2^2 \cup \left(a,0,...,0\right)2^1 \cup \left(n_0=1\right)$$

will give a SOSRD under intra-class correlated structure of errors using PBIB type designs in N= 57 design points for six factors.

Here we have,

$$\sum_{u=1}^{N} X_{iu}^2 = 21.6569+2a^2 = N\mu_2 \tag{3.5}$$

$$\sum_{u=1}^{N} X_{iu}^4 = 24+2a^4 = cN\mu_4 \tag{3.6}$$

$$\sum_{u=1}^{N} X_{iu}^2 X_{ju}^2 = 8 = N\mu_4 \tag{3.7}$$

Substituting for μ_2, μ_4, A= {1+ (N-1) ρ} and c in (3.1) and on simplification, we get the following fourth degree polynomial equation in a^2.

$$\left(1+56\rho\right)^2\left(90a^8-519.764502a^6+49.883984a^4+2772.077344a^2-6879.381249\right)=0 \tag{3.8}$$

Equation (3.8) has only one positive real root a^2 = 5.0514 $((\forall(\frac{1}{N-1}<\rho<1)))$. This can be alternatively written directly from (3.1). By substituting this 'a' value in (3.5), (3.6) and (3.7) on simplification we get $\mu_2=0.5572,\ \mu_4=0.1404$, c = 9.379118. From (2.14) non-singularity condition (0.4522 > 0.4175) is also satisfied. From (2.7) to (2.12), we obtain the variances and covariances. Further, from (2.19), we get,

$$V\left(\frac{\partial\hat{y}}{\partial x_i}\right)=\left(0.0283+0.1125d^2\right)(1-\rho)\sigma^2. \tag{3.9}$$

4 A Study on Dependence of the Variance Function of the Response at Different Design Points

Here, we study the dependence of variance function of response at different designs points of SOSRD under intra-class correlated structure of errors using PBIB type designs. Given 'v' factors different values of intra-class correlation coefficient ρ and distance from centre'd' (centre of the design) between 0 and 1, the variances are tabulated. From (3.9) the variance of the estimated derivative is obtained by

$$V\left(\frac{\partial\hat{y}_u}{\partial X_i}\right)=0.0294\ \left(\text{taking }\rho=0.1,\ \sigma=1,\ d=0.1\right).$$

The numerical calculations are provided in Tables 1 and 2.

Table 2. Study of dependence of estimated slope of second order response surface design under intra-class correlated structure of errors PBIB type designs at different design points for $6\le v\le 12$ factors for different values of 'ρ', 'd' and σ=1 (v=6, b_1=4, r_1=2, k_1=3, λ_1=1, λ_2=0) , (v=6, b_2=3, r_2=6, k_2=2, λ_1'=0, λ_2'=1)

ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0327	0.0365	0.0427	0.0515	0.0627	0.0765	0.0927	0.1115	0.1327	0.1565
0.1	0.0294	0.0328	0.0384	0.0463	0.0564	0.0688	0.0834	0.1003	0.1194	0.1408
0.2	0.0262	0.0292	0.0342	0.0412	0.0502	0.0612	0.0742	0.0892	0.1062	0.01252
0.3	0.0228	0.0255	0.0298	0.036	0.0438	0.0535	0.0648	0.078	0.0928	0.1095
0.4	0.0197	0.0219	0.0257	0.0309	0.0377	0.0459	0.0557	0.0669	0.0797	0.0939
0.5	0.0163	0.0182	0.0213	0.0257	0.0313	0.0382	0.0463	0.0557	0.0663	0.0782
0.6	0.0131	0.0146	0.0171	0.0206	0.0251	0.0306	0.0371	0.0446	0.0531	0.0626
0.7	0.0098	0.0109	0.0128	0.0154	0.0188	0.0229	0.0278	0.0334	0.0398	0.0469
0.8	0.0066	0.0073	0.0086	0.0103	0.0126	0.0153	0.0186	0.0223	0.0266	0.0313
0.9	0.0033	0.0037	0.0043	0.0052	0.0063	0.0077	0.0093	0.0112	0.0133	0.0157

Table 3. Comparison of different methods of construction of SOSRD under intra-class correlated structure of errors

v	CCD [16]	BIBD [18]	PBD [15]	SUBA with two unequal block sizes [17]	A Pair of BIBD [21]	PBIBD
2	9	-	-	-	-	-
3	15	19 (3, 3, 2, 2, 1)	-	-	-	-
4	25	33 (4, 6, 3, 2, 1)	-	-	-	-
5	27	51 (5, 10, 4, 2, 1)	-	-	-	-
6	45	73 (6, 15, 5, 2, 1)	69 (6, 7, 3, 2, 3, 4, 1)	69 (6, 7, 3, 3, 2, 1)	-	**57** (6, 4, 2, 3, 1, 0) (6, 3, 1, 2, 0, 1)
7	79	71 (7, 7, 3, 3, 1)	-	-	141 (7, 7, 3, 3, 1) (7, 21, 6, 2, 1)	-
8	81	129 (8, 28, 7, 2, 1)	257 (8, 15, 6, 4, 3, 2, 2)	113 (8, 15, 4, 2, 3, 4, 8, 1)	337 (8, 14, 7, 3, 2) (8, 28, 7, 2, 1)	**97** (8, 8, 3, 3, 1, 0) (8, 4, 1, 2, 0, 1)
9	147	115 (9, 12, 4, 3, 1)	195 (9, 11, 5, 5, 4, 3, 2)	163 (9, 18, 5, 4, 3, 9, 9, 1)	241 (9, 12, 4, 3, 1) (9, 36, 8, 2, 1)	127 (9, 9, 3, 3, 1, 0) (9, 9, 2, 2, 0, 1)
10	149	201 (10, 45, 9, 2, 1)	197 (10, 11, 5, 5, 4, 2)	197 (10, 11, 5, 4, 5, 5, 6, 2)	421 (10, 15, 6, 4, 2) (10, 45, 9, 2, 1)	**169** (10, 8, 4, 5, 2, 0) (10, 5, 1, 2, 0, 1)
11	151	199 (11, 11, 5, 5, 2)	-	-	391 (11, 11, 5, 5, 2) (11, 55, 10, 2, 1)	-
12	281	377 (12, 44, 11, 3, 2)	537 (12, 16, 6, 6, 5, 4, 3, 2)	233 (12, 13, 4, 3, 4, 4, 9, 1)	881 (11, 11, 5, 5, 2) (11, 55, 10, 2, 1)	**305** (12, 8, 4, 6, 2, 0) (12, 6, 1, 2, 0, 1)
13	283	235	539	-	417	-

v	CCD [16]	BIBD [18]	PBD [15]	SUBA with two unequal block sizes [17]	A Pair of BIBD [21]	PBIBD
		(13, 13, 4, 4, 1)	(13, 16, 6, 6, 5, 4, 3, 2)		(13, 13, 4, 4, 1) (13, 26, 6, 3, 1)	
14	285	-	541 (14, 16, 6, 6, 5, 4, 2)	309 (14, 35, 7, 2, 3, 7, 28, 1)	-	-
15	287	311 (15, 35, 7, 3, 1)	543 (15, 16, 6, 6, 5, 2)	351 (15, 20, 5, 3, 4, 5, 15, 1)	1241 (15, 15, 7, 7, 1) (15, 35, 7, 3, 1)	-
16	289	353 (16, 20, 5, 4, 1)	-	481 ((16, 28, 6, 4, 3, 12, 16, 1)	1151 (16, 16, 6, 6, 2) (16, 80, 15, 3, 2)	-

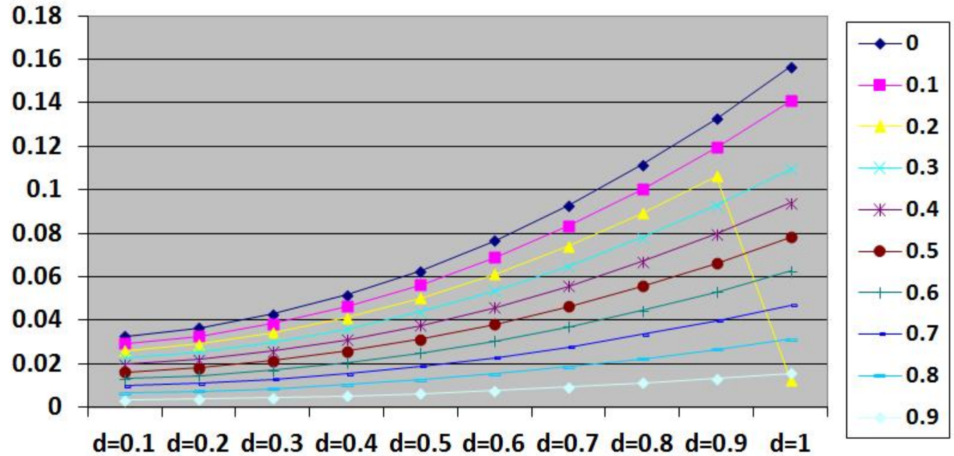

Fig. 1. The graphical representation of study of dependence of estimated slope of second order response surface design under intra-class correlated structure of errors PBIB type designs at different design points for v = 6 factors for different values of ‘ ρ ’, ‘d’ and σ=1. (v=6, b_1=4, r_1=2, k_1=3, λ_1=1, λ_2=0), (v=6, b_2=3, r_2=6, k_2=2, λ_1'=0, λ_2'=1)

5 Conclusions

From Table 1, 2 and 3 we observe that,

1. When the values of “ρ” is increases slope rotatability value of “a” is decreases for all the v factors.
2. At ρ = 0 estimated value and slope rotatability derivative of SOSRD under tri-diagonal correlated structure is equal to the SOSRD uncorrelated errors case.
3. In this method, we obtain designs with fewer number of design points in some cases. The implications of fewer number of design points leads to effective and reduced cost of experimentation.

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Competing Interests

Authors have declared that no competing interests exist.

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