Asian Research Journal of Mathematics

Volume 20, Issue 10, Page 15-29, 2024; Article no.ARJOM.122295
ISSN: 2456-477X

S-Prime Graph of S-meet Semilattice

Mythily C V [a*] and Kalamani D [a]

[a] *Department of Mathematics, Bharathiar University PG Extension and Research centre, Erode, Tamil Nadu, India.*

Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information

DOI: https://doi.org/10.9734/arjom/2024/v20i10841

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/122295

Received: 07/07/2024
Accepted: 11/09/2024
Published: 19/09/2024

Original Research Article

Abstract

In this paper, the S-prime ideal in a lattice $\mathfrak{L}$ is introduced where S is the meet subset of $\mathfrak{L}$. Also, it is shown that the prime ideal of $\mathfrak{L}$ is an S-prime ideal of $\mathfrak{L}$ and studied with suitable examples. Further, the S-prime ideal $\mathfrak{I}_{\mathfrak{s}}$ of S-meet semilattice L_s is introduced. Finally, a new graph called S-prime graph of S-meet semilattice is defined and their topological measures are generalized.

Keywords: Ideal of a lattice; lattice; partially ordered set; prime ideal; semilattice; S-prime ideal.

2010 Mathematics Subject Classification: 13A15; 03G10; 68R10.

**Corresponding author: E-mail: mythilyvaradharajan@gmail.com;*

Cite as: *C V, Mythily, and Kalamani D. 2024. "S-Prime Graph of S-Meet Semilattice". Asian Research Journal of Mathematics 20 (10):15-29. https://doi.org/10.9734/arjom/2024/v20i10841.*

1 Introduction

In 1961, Gratzer and Schmidt [1] defined a standard ideal in $\mathfrak{L}$ and Noor and Latif [2] introduced and discussed about the standard n-ideal of $\mathfrak{L}$. In 1994, the n-ideals in $\mathfrak{L}$ were introduced by Latif and Noor [3]. After that, they studied finitely generated n-ideals of $\mathfrak{L}$ [4]. In 2000, the properties of standard n-ideal of $\mathfrak{L}$ were discussed by Noor and Latif [5].

In 2015, Meenakshi P and Karuna T [6] introduced the 2-absorbing and weakly 2-absorbing ideals of $\mathfrak{L}$ which was from [7 - 8]. A proper ideal I of $\mathfrak{L}$ is called a 2-absorbing ideal if $a \wedge b \wedge c$ is in I for a, b, c is in $\mathfrak{L}$ then either $a \wedge b$ or $a \wedge c$ or $b \wedge c$ is in I. Also, they defined the triple zero in lattices and given some results related to triple zero. In 2021, Ali Akbar and Toktam Haghdadi [9], introduced the n-absorbing ideals in $\mathfrak{L}$ which is from [10]. Many authors have introduced and studied different ideals in a lattice, such as: semiprime n-ideal of $\mathfrak{L}$ [11], modular n-ideals of $\mathfrak{L}$ [12] and so on.

In 2019, Ahmed Hamed and Achraf Malek [13] defined S-prime ideals of $\mathfrak{R}$. A proper ideal I of $\mathfrak{R}$ is called an S-prime ideal I_s of $\mathfrak{R}$ if x, y is in $\mathfrak{R}$ and xy is in I_s then sx or sy is in I_s for some $s \in S$ where S is the multiplicative subset of $\mathfrak{R}$. The multiplicative subset is the complement of the prime ideal of a ring $\mathfrak{R}$.

Recently, Kalamani and Mythily [14] introduced a graph called S-prime ideal graph in which the vertices of the graph are elements of $\mathfrak{R}$ and they are connected iff $sa \in I_s$ or $sb \in I_s$ for some $s \in S$ whenever $ab \in I_s$ where $a, b \in \mathfrak{R}$ and the set S is disjoint from I_s. Some of the properties of the S-prime ideal I_s of $\mathfrak{R}$ are discussed in [15] and they [16] studied the interplay of the semilattice theoretic properties of a poset with the ring theoretic properties.

In this article, the S-prime ideal of $\mathfrak{R}$ is defined in a lattice $\mathfrak{L}$ and in a S-meet semilattice L_s and some results are discussed. Also, a new graph called the S-prime graph is defined and their topological measures are generalized. Refer [17 - 19] for background research related to the indices.

Throughout this paper, the first and second Zagreb indices and the Randi´c index of $\mathfrak{G}(\mathfrak{I}_s)$ are denoted $M_1(\mathfrak{G}(\mathfrak{I}_s)), M_2(\mathfrak{G}(\mathfrak{I}_s))$ and $R(\mathfrak{G}(\mathfrak{I}_s))$ respectively.

This article is organized as follows: Section 2 recalls some basic notions and definitions of lattice theory and topological indices of a graph. In section 3, the definitions of meet subset and S-prime ideal of a lattice are given with suitable examples. In section 4, the S-prime ideal of S-meet semilattice are introduced. Also, a new graph called the S-prime graph of S-meet semilattice is introduced with suitable examples. Some topological measures of the S-prime graph are discussed in sections 5, 6 and 7.

2 Preliminaries

In this section, the necessary definitions are recalled from [17 - 20].

Definition 2.1. *A relation $\mathcal{R}$ on a set A is said to be partial order relation if the relation $\mathcal{R}$ is reflexive, antisymmetric and transitive which may be described as follows: 1) Reflexivity: $a \sim a$ for all $a \in A$. 2) Antisymmetry: If $a \sim b$ and $b \sim a$ then $a = b$. 3) Transitivity: If $a \sim b$; $b \sim c$ then $a \sim c$. A set together with the partial order relation $\mathcal{R}$ is called poset.*

Definition 2.2. *A **lattice** $\mathfrak{L}$ is a poset in which every a, b in $\mathfrak{L}$ has meet $(\wedge)$ and join $(\vee)$. It is denoted as $(\mathfrak{L}, \wedge, \vee)$.*

Definition 2.3. *Let $(\mathfrak{L}, \wedge, \vee)$ be a lattice and $M \subseteq \mathfrak{L}$. Then $(M, \wedge, \vee)$ is a **sublattice** of $(\mathfrak{L}, \wedge, \vee)$ iff M is closed under $\wedge$ and $\vee$.*

Definition 2.4. *The sublattice I of $\mathfrak{L}$ is an **ideal** of $\mathfrak{L}$ if $a \wedge i \in I$ for every $i \in I$ and $a \in \mathfrak{L}$.*

Definition 2.5. *The sublattice I of $\mathfrak{L}$ is **prime ideal** of $\mathfrak{L}$ if $a \wedge b \in I$ implies $a \in I$ or $b \in I$ for every $a, b \in \mathfrak{L}$.*

Definition 2.6. *The topological measures of the graph $\mathfrak{G}$ are defined as follows:*

*The **first Zagreb index of a graph** $\mathfrak{G}$ is*

$$M_1(\mathfrak{G}) = \sum_{\mathfrak{x}\in\mathcal{V}(\mathfrak{G})} \mathfrak{d}(\mathfrak{x})^2.$$

*The **second Zagreb index of a graph** $\mathfrak{G}$ is*

$$M_2(\mathfrak{G}) = \sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}(G)} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}).$$

*The **first Zagreb coindex of a graph** $\mathfrak{G}$ is*

$$\overline{M_1}(\mathfrak{G}) = \sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G})}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})].$$

*The **second Zagreb coindex of a graph** $\mathfrak{G}$ is*

$$\overline{M_2}(\mathfrak{G}) = \sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G})} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}).$$

*The **Randi ́c index of a graph** $\mathfrak{G}$ is*

$$R(\mathfrak{G}) = \sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}(\mathfrak{G})} \frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}}.$$

3 S-prime Ideal of a Lattice

In this section, the S-prime ideal of $\mathfrak{L}$ is defined with an example and some of its results are discussed.

Definition 3.1. *Let $S \subseteq \mathfrak{L}$. Then the set S is called meet subset of $\mathfrak{L}$ if $a \wedge b \in S$ for all $a, b \in S$.*

Definition 3.2. *Let I be a proper ideal of a lattice $\mathfrak{L}$. The ideal I is said to be an S-prime ideal of $\mathfrak{L}$ if $x \wedge y$ in I then $s \wedge x$ or $s \wedge y$ is in I for any $x, y \in \mathfrak{L}$ and for some $s \in S$, where S is the meet subset of a lattice $\mathfrak{L}$ which is disjoint from I of $\mathfrak{L}$. The S-prime ideal of $\mathfrak{L}$ is denoted by I_s*

Example 3.3. *Consider $\mathfrak{L} = \{0, u, v, w, x, y, z, 1\}$ be a lattice whose Hasse diagram is given in Fig. 1. The S-prime ideal I_s of $\mathfrak{L} = \{0, u, v, w, x, y, z, 1\}$ are $I_1 = \{0\}, I_2 = \{0, u\}, I_3 = \{0, u, v\}, I_4 = \{0, u, w\}, I_5 = \{0, u, x\}, I_6 = \{0, u, y\}$ and $I_7 = \{0, u, v, w, x, y, z\}$ from Fig. 1. The meet subset of a lattice are $S_1 = \{1\}, S_2 = \{1, u\}, S_3 = \{1, v\}, S_4 = \{1, w\}, S_5 = \{1, x\}, S_6 = \{1, y\}, S_7 = \{1, z\}, S_8 = \{1, v, z\}, S_9 = \{1, v, z\}$, $S_{10} = \{1, v, z\}$ and so on.*

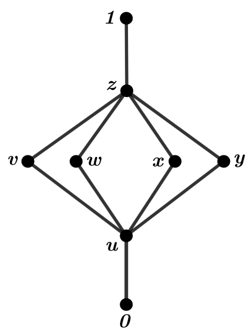

Fig. 1. Hasse diagram of $\mathfrak{L}$

Theorem 3.4. *Every prime ideal P of $\mathfrak{L}$ is an S-prime ideal of $\mathfrak{L}$.*

Proof. Let P be the prime ideal of $\mathfrak{L}$.
Let S be the meet subset of $\mathfrak{L}$ which is disjoint from P of $\mathfrak{L}$. That is, $P \cap S = \emptyset$.
Choose $x, y \in \mathfrak{L}$ such that $x \wedge y \in P$.
Since P is prime, x or $y \in P$.
If $s \in S$ then $s \wedge x$ or $s \wedge y \in P$.
Thus, the prime ideal P is the S-prime ideal I_s of $\mathfrak{L}$. □

The converse of Theorem 3.4 is not true and is explained in the following example.

Example 3.5. *Let us consider the example which is shown in Fig.1. Let $I_s = \{0, u\}$ be the S-prime ideal of $\mathfrak{L} = \{0, u, v, w, x, y, z, 1\}$ and the meet subset of $\mathfrak{L}$ as $S = \{1, z, y\}$. Now, let $v, w \in \mathfrak{L}$ if $v \wedge w = u \in I_s$ which implies that $v \notin I_s$ and $w \notin I_s$. Thus, I_s is not the prime ideal of $\mathfrak{L}$.*

Theorem 3.6. *Let P be the prime ideal of $\mathfrak{L}$. Then $\mathfrak{L} - P$ is the meet subset of $\mathfrak{L}$.*

Proof. Let $\mathfrak{L}$ be the lattice and P be the prime ideal of $\mathfrak{L}$. It is needed to prove that the set $\mathfrak{L} - P$ is a meet subset of $\mathfrak{L}$. Let $x, y \in \mathfrak{L} - P$.
This implies that $x, y \in \mathfrak{L}$ and $x, y \notin P$.
Suppose $x \wedge y \in P$. As P is prime, either x or y is in P.
This contradicts to $x, y \notin P$. Therefore, $x \wedge y \notin P$.
The elements $x, y \in \mathfrak{L}$ implies that $x \wedge y \in \mathfrak{L}$. Therefore, $x \wedge y \in \mathfrak{L} - P$.
Thus, the set $\mathfrak{L} - P$ is the meet subset of a lattice $\mathfrak{L}$. □

4 S-prime Graph of an S-meet Semilattice

In this section, the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ of an S-meet semilattice L_s is defined. Also, a new graph called the S-prime graph of L_s is explained with necessary examples. The concept of the meet subset and the S-prime ideal of $\mathfrak{L}$ are applicable to the S-meet semilattice L_s of a ring $\mathfrak{R}$.

Definition 4.1. *Let I be the proper ideal of L_s. The ideal I is said to be an S-prime ideal $\mathfrak{I}_\mathfrak{s}$ of L_s if for any $\mathfrak{u}, \mathfrak{v} \in L_s, \mathfrak{u} \wedge \mathfrak{v} \in \mathfrak{I}_\mathfrak{s}$ then $\exists\ s \in S$ such that $s \wedge \mathfrak{u}$ or $s \wedge \mathfrak{v}$ in $\mathfrak{I}_\mathfrak{s}$ for some $s \in S$, where S is the meet subset of L_s and $S \cap \mathfrak{I}_\mathfrak{s} = \emptyset$.*

Definition 4.2. *Let $(L_s, \wedge, \subseteq)$ be the S-meet semilattice where L_s is the collection of all S-prime ideals of $\mathfrak{R}$. The set of all elements of L_s are considered to be the vertices of the graph, the vertices $\mathfrak{x}$ and $\mathfrak{y}$ are adjacent if $\mathfrak{x} \wedge \mathfrak{y} \in \mathfrak{I}_\mathfrak{s}$, where $\mathfrak{I}_\mathfrak{s}$ is the S-prime ideal of L_s. It is an undirected graph called S-prime graph of the S-prime ideal $\mathfrak{I}_\mathfrak{s}$, denoted by $\mathfrak{G}_{L_s}(\mathfrak{I}_\mathfrak{s})$, simply $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$.*

Let $\mathfrak{R}$ be a ring of order $p^t q$. The S-prime graph $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ of $\mathfrak{I}_\mathfrak{s}$ is (i) a complete graph if the S-prime ideals $\mathfrak{I}_\mathfrak{s}$ of L_s are $\downarrow p, \downarrow q$ and $\downarrow pq$, (ii) a star graph if the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ of L_s is $\downarrow p^t q$ and (iii) a connected graph if the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ of L_s is $\downarrow p^k q, k < t$.

Example 4.3. *Let $\mathfrak{R} = \mathbb{Z}_{48}$ and the S-prime graphs $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ are shown in Fig. 2. The elements of L_s are $< 2 >, < 3 >, < 6 >, < 12 >, < 24 >$ and $< 48 >$.*

Let $\mathfrak{R}$ be a ring of order pqr. Then the S-prime graph $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ is a complete graph if the S-prime ideals of L_s are $\downarrow p \cup \downarrow q, \downarrow p \cup \downarrow r, \downarrow q \cup \downarrow r, \downarrow pq \cup \downarrow pr \cup \downarrow qr, \downarrow p \cup \downarrow qr, \downarrow q \cup \downarrow pr$ and $\downarrow r \cup \downarrow pq$.

There are 3 distinct connected S-prime graphs $\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s}), \mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})$ and $\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})$ where $\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})$ is the S-prime graph for the S-prime ideals $\downarrow p$, $\downarrow q$, $\downarrow r$, $\downarrow pq \cup \downarrow pr$, $\downarrow pq \cup \downarrow qr$, $\downarrow pr \cup \downarrow qr$, $\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})$ is the S-prime graph for the S-prime ideals $\downarrow pq$, $\downarrow pr$, $\downarrow qr$ and $\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})$ for the S-prime ideal $\downarrow pqr$.

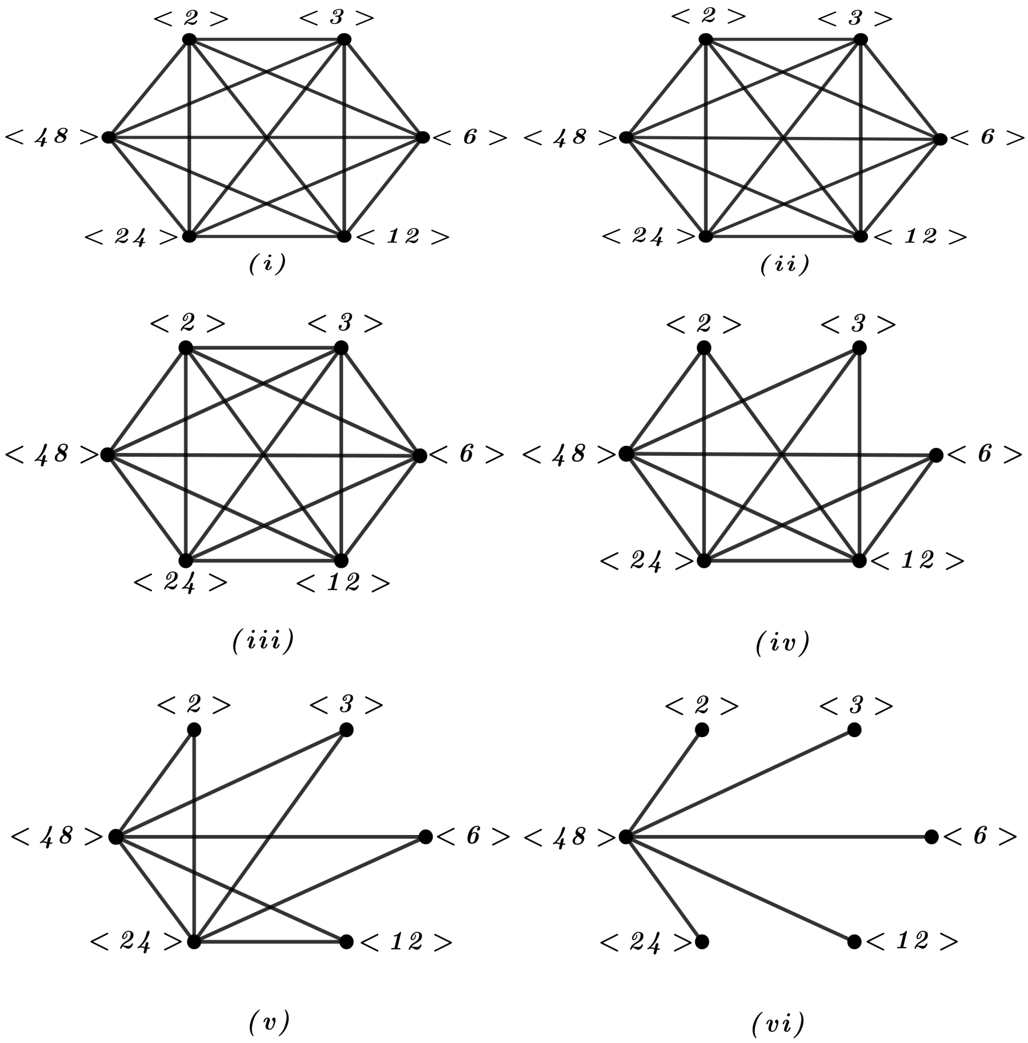

Fig. 2. **S-prime graph of the S-prime ideal** $(i) \downarrow 2$ $(ii) \downarrow 3$ $(iii) \downarrow 6$ $(iv) \downarrow 12$ **(v)**$\downarrow 24$ $(vi) \downarrow 48$

Example 4.4. *Let $\mathfrak{R} = \mathbb{Z}_{30}$ and the S-prime graph of S-meet semilattice is shown in Fig. 3 whose vertex set is*

$\mathcal{V}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) = \{< 2 >, < 3 >, < 5 >, < 6 >, < 10 >, < 15 >, < 30 >\}.$

In the following sections, the topological measures $M_1(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$, $M_2(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$, $\overline{M_1}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$, $\overline{M_2}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$ and $R(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$ of the connected S-prime graph of the S-prime ideals $\mathfrak{I}_{\mathfrak{s}}$ are studied.

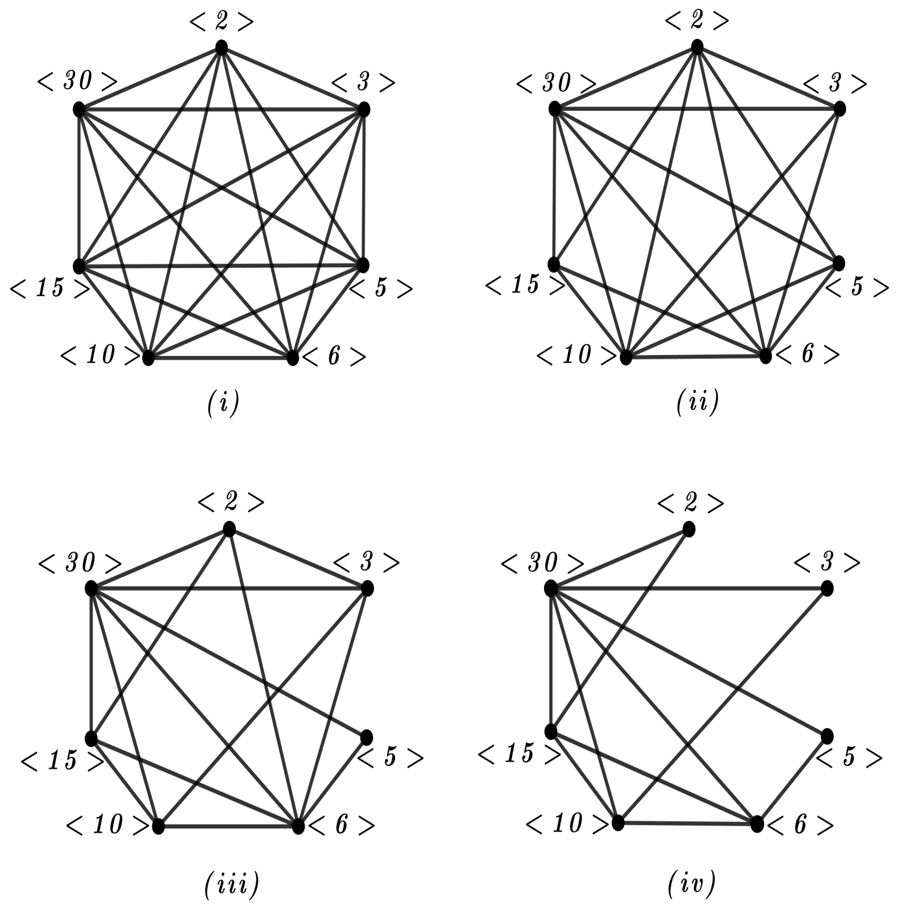

Fig. 3. ***S*-prime graph of the *S*-prime ideal** $(i) \downarrow 2 \cup \downarrow 3$ $(ii) \downarrow 2$ $(iii) \downarrow 6$ $(iv) \downarrow 30$

5 First and Second Zagreb Indices of S-prime Graph

Let $\mathfrak{R}$ be a ring of order $p^t q$. The S-prime graph $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ is connected if the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ is the down-set of $p^k q$ where $k < t$. Let ϖ and ϑ be the order of the graph $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ and the ideal $\mathfrak{I}_\mathfrak{s}$ respectively.

Theorem 5.1. *Let $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ be the S-prime graph of the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ of L_s then*

$$M_1(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})) = \vartheta\Big[(\varpi - 1)^2 + (\varpi - \vartheta)\vartheta\Big].$$

Proof. Let $\mathfrak{x}$ be a vertex of $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$.

If $\mathfrak{x}$ is an element of the S-prime ideal $\mathfrak{I}_\mathfrak{s}$, then $\mathfrak{x} \wedge \mathfrak{y} \in \mathfrak{I}_\mathfrak{s}\ \forall\ \mathfrak{y} \in L_s$.

If $\mathfrak{x}$ is not an element of $\mathfrak{I}_\mathfrak{s}$, then $\mathfrak{x} \wedge \mathfrak{y} \in \mathfrak{I}_\mathfrak{s}$ only if $\mathfrak{y} \in \mathfrak{I}_\mathfrak{s}$.

Therefore, $\mathfrak{d}(\mathfrak{x})$ is given as follows:

$$\mathfrak{d}(\mathfrak{x}) = \begin{cases} \varpi - 1 \ \ if \ \ \mathfrak{x} \in \mathfrak{I}_{\mathfrak{s}} \\ \vartheta \ \ otherwise. \end{cases} \tag{5.1}$$

Then, $M_1(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) = \sum_{\mathfrak{x}\in\mathcal{V}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))} \mathfrak{d}(\mathfrak{x})^2$

$= \sum_{\mathfrak{x}\in\mathfrak{I}_{\mathfrak{s}}} \mathfrak{d}(\mathfrak{x})^2 + \sum_{\mathfrak{x}\notin\mathfrak{I}_{\mathfrak{s}}} \mathfrak{d}(\mathfrak{x})^2$

$= \vartheta(\varpi - 1)^2 + (\varpi - \vartheta)\vartheta^2$

$= \vartheta\Big[(\varpi - 1)^2 + (\varpi - \vartheta)\vartheta\Big].$ □

Theorem 5.2. *Let $\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})$ be the S-prime graph of the S-prime ideal $\mathfrak{I}_{\mathfrak{s}}$ of L_s then*

$$M_2(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) = \vartheta(\varpi - 1)\Big[\frac{(\vartheta-1)}{2}(\varpi - 1) + \vartheta(\varpi - \vartheta)\Big].$$

Proof. Let $\mathcal{E}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$ be the edge set of $\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})$ of the S-prime ideal $\mathfrak{I}_{\mathfrak{s}}$ of L_s. Let $\mathfrak{x}\mathfrak{y} \in \mathcal{E}[\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})]$.

This implies that either $\mathfrak{x}$ or $\mathfrak{y}$ is in $\mathfrak{I}_{\mathfrak{s}}$ and $\mathfrak{d}(\mathfrak{x})$ and $\mathfrak{d}(\mathfrak{y})$ are defined in (5.1). Then,

$M_2(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) = \sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})$

$= \sum_{\mathfrak{x}\in\mathfrak{I}_{\mathfrak{s}},\mathfrak{y}\in\mathfrak{I}_{\mathfrak{s}}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}) + \sum_{\mathfrak{x}\in\mathfrak{I}_{\mathfrak{s}},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})$

$= \vartheta\dfrac{(\vartheta-1)}{2}(\varpi - 1)^2 + \vartheta(\varpi - \vartheta)(\varpi - 1)\vartheta$

$= \vartheta(\varpi - 1)\Big[\dfrac{(\vartheta-1)}{2}(\varpi - 1) + \vartheta(\varpi - \vartheta)\Big].$ □

Let $\mathfrak{R}$ be a ring of order pqr. There are 3 distinct S-prime connected graphs of $\mathfrak{I}_{\mathfrak{s}}$ namely $\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}}), \mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})$ and $\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})$ which are explained earlier.

Theorem 5.3. *Let $\mathfrak{R}$ be a ring of order pqr. Then,*

(i) $M_1(\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}})) = \vartheta\Big[(\varpi - 1)^2 + \vartheta(\varpi - \vartheta)\Big].$

(ii) $M_1(\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})) = \vartheta\Big[(\varpi - 1)^2 + \vartheta)\Big] + (\vartheta + 2)^2.$

(iii) $M_1(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})) = \vartheta(\varpi - 1)^2 + (\varpi - 4)\Big[(\vartheta + 1)^2 + (\vartheta + 3)^2\Big].$

Proof. (i) Let $\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}})$ be the S-prime graph of $\mathfrak{I}_{\mathfrak{s}}$ of L_s. Then,

$M_1(\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}})) = \sum_{\mathfrak{x}\in\mathcal{V}(\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}}))} \mathfrak{d}(\mathfrak{x})^2$

$= \sum_{\mathfrak{x}\in\mathfrak{I}_{\mathfrak{s}}} \mathfrak{d}(\mathfrak{x})^2 + \sum_{\mathfrak{x}\notin\mathfrak{I}_{\mathfrak{s}}} \mathfrak{d}(\mathfrak{x})^2$

$= \vartheta(\varpi-1)^2 + (\varpi-\vartheta)\vartheta^2$

$= \vartheta\Big[(\varpi-1)^2 + \vartheta(\varpi-\vartheta)\Big].$

(ii) Let $\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})$ be the S-prime graph of $\mathfrak{I}_\mathfrak{s}$ of L_s.

In this case, the non-ideal elements are adjacent to all the ideal elements and some non-ideal elements. Here, the S-prime ideals are $\downarrow pq$, $\downarrow pr$ and $\downarrow qr$.

Consider the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ as $\downarrow pq$ and $\mathfrak{d}(\mathfrak{x})$ in $\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})$ is as follows:

$$\mathfrak{d}(\mathfrak{x}) = \begin{cases} \varpi-1 \;\; if \;\; \mathfrak{x}\in\mathfrak{I}_\mathfrak{s} \\ \vartheta \;\; if \;\; \mathfrak{x}=r \\ \vartheta+2 \;\; otherwise. \end{cases} \tag{5.2}$$

Then,

$M_1(\mathfrak{G}_s^{(2)}(I)) = \sum_{\mathfrak{x}\in\mathcal{V}(\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s}))} \mathfrak{d}(\mathfrak{x})^2$

$= \sum_{\mathfrak{x}\in\mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})^2 + \sum_{\mathfrak{x}=r} \mathfrak{d}(\mathfrak{x})^2 + \sum_{\mathfrak{x}\neq r} \mathfrak{d}(\mathfrak{x})^2$

$= \vartheta(\varpi-1)^2 + \vartheta^2 + (\vartheta+2)^2$

$= \vartheta\Big[(\varpi-1)^2 + \vartheta)\Big] + (\vartheta+2)^2.$

(iii) Let $\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})$ be the S-prime graph of $\mathfrak{I}_\mathfrak{s}$ of L_s.

In this case, the S-prime ideal is $\downarrow pqr$. The maximal elements of L_s are p, q, r and they are denoted as $\mathfrak{M}_k, k = 1, 2, 3$ and $\mathfrak{d}(\mathfrak{x})$ in $\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})$ is as follows:

$$\mathfrak{d}(\mathfrak{x}) = \begin{cases} \varpi-1 \;\; if \;\; \mathfrak{x}\in\mathfrak{I}_\mathfrak{s} \\ \vartheta+1 \;\; if \;\; \mathfrak{x}=\mathfrak{M}_k \\ \vartheta+3 \;\; otherwise. \end{cases} \tag{5.3}$$

Then,

$M_1(\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x}\in\mathcal{V}(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s}))} \mathfrak{d}(\mathfrak{x})^2$

$= \sum_{\mathfrak{x}\in\mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})^2 + \sum_{\mathfrak{x}=\mathfrak{M}_k} \mathfrak{d}(\mathfrak{x})^2 + \sum_{\mathfrak{x}\neq\mathfrak{M}_k} \mathfrak{d}(\mathfrak{x})^2$

$= \vartheta(\varpi-1)^2 + (\varpi-4)(\vartheta+1)^2 + (\varpi-4)(\vartheta+3)^2$

$= \vartheta(\varpi-1)^2 + (\varpi-4)\Big[(\vartheta+1)^2 + (\vartheta+3)^2\Big].$ □

Theorem 5.4. *Let $\mathfrak{R}$ be a ring of order pqr. Then,* *(i)* $M_2(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})) = (\varpi - 1)\Big[(\varpi - 1)^2 + \vartheta^2(\varpi - \vartheta)\Big].$

(ii) $M_2(\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})) = (\varpi - 1)\Big[(\varpi - 1) + \vartheta^2\Big] + (\vartheta + 2)\Big[8(\vartheta - 1) + (\vartheta + 2)^2\Big].$

(iii) $M_2(\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})) = 2(\varpi - 4)\Big[(\varpi - 1)(\eta - 4) + (\vartheta + 3)(\varpi + 2)\Big].$

Proof. (i) Let $\mathcal{E}[\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})]$ be the edge set of $\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})$ of the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ of L_s. Let $\mathfrak{x}\mathfrak{y} \in \mathcal{E}[\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})]$. This implies that either $\mathfrak{x}$ or $\mathfrak{y}$ is in $\mathfrak{I}_\mathfrak{s}$ and the degrees of the vertices $\mathfrak{x}$ and $\mathfrak{y}$ are defined in (5.1). Then,

$$\begin{aligned}
M_2(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})) &= \textstyle\sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s}))} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}) \\
&= \textstyle\sum_{\mathfrak{x},\mathfrak{y}\in\mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}) + \sum_{\mathfrak{x}\in\mathfrak{I}_\mathfrak{s},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}) \\
&= (\varpi - 1)(\varpi - 1)^2 + \vartheta(\varpi - \vartheta)(\varpi - 1)\vartheta \\
&= (\varpi - 1)\Big[(\varpi - 1)^2 + \vartheta^2(\varpi - \vartheta)\Big].
\end{aligned}$$

(ii) Let $\mathcal{E}[\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})]$ be the edge set of $\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})$ of the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ of L_s.
Consider the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ as $\downarrow pq$ and $\mathfrak{d}(\mathfrak{x}), \mathfrak{d}(\mathfrak{y})$ of $\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})$ are defined in (5.2). Then,

$$\begin{aligned}
M_2(\mathfrak{G}_s(\mathfrak{I}_\mathfrak{s})) &= \textstyle\sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}[\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})]} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}) \\
&= \textstyle\sum_{\mathfrak{x},\mathfrak{y}\in\mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}) + \sum_{\mathfrak{x}\in\mathfrak{I}_\mathfrak{s},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}) + \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}) \\
&= (\varpi - 1)^2 + \Big[(\varpi - 1)\vartheta^2 + 8(\varpi - 1)(\vartheta + 2)\Big] + (\vartheta + 2)^3 \\
&= (\varpi - 1)\Big[(\varpi - 1) + \vartheta^2\Big] + (\vartheta + 2)\Big[8(\varpi - 1) + (\vartheta + 2)^2\Big].
\end{aligned}$$

(iii) Let $\mathcal{E}[\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})]$ be the edge set of $\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})$ of the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ of L_s.
Consider the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ as $\downarrow pqr$ and $\mathfrak{d}(\mathfrak{x}), \mathfrak{d}(\mathfrak{y})$ of $\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})$ are defined in (5.3). Then,

$$\begin{aligned}
M_2(\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})) &= \textstyle\sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}[\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})]} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}) \\
&= \textstyle\sum_{\mathfrak{x}\in\mathfrak{I}_\mathfrak{s},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})} + \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}) \\
&= \Big[(\varpi - 1)(\varpi - 5)(\varpi - 4) + (\varpi - 1)(\varpi - 3)(\varpi - 4)\Big] + \Big[(\vartheta + 1)(\vartheta + 3)(\varpi - 4) + (\vartheta + 3)^2(\varpi - 4)\Big] \\
&= (\varpi - 1)(\varpi - 4)(2\varpi - 8) + (\varpi - 4)(\vartheta + 3)(2\varpi + 4) \\
&= 2(\varpi - 1)(\varpi - 4)^2 + 2(\varpi - 4)(\vartheta + 3)(\varpi + 2) \\
&= 2(\varpi - 4)\Big[(\varpi - 1)(\varpi - 4) + (\vartheta + 3)(\varpi + 2)\Big].
\end{aligned}$$

□

6 First and Second Zagreb Coindex of S-prime Graph

The $M_1(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))$ and $M_2(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))$ of the S-prime graph are generalized in this section.

Theorem 6.1. *Let $\mathfrak{R}$ be a ring of order $p^t q$. Then,*

$$\overline{M_1}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})) = \vartheta(\varpi - \vartheta)(\varpi - \vartheta - 1).$$

Proof. Let $\mathfrak{x}\mathfrak{y} \in \mathcal{E}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))$ be the edge set of $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ of the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ of L_s. If $\mathfrak{x}\mathfrak{y}$ is an edge of $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$, then at least one of the end points of $\mathfrak{x}\mathfrak{y}$ must be in the ideal $\mathfrak{I}_\mathfrak{s}$. Then,

$\overline{M_1}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x}\mathfrak{y} \notin \mathcal{E}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))} [\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})]$

In this, there is no edge between the non-ideal elements $\mathfrak{x}, \mathfrak{y}$. Thus,

$\overline{M_1}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x},\mathfrak{y} \notin \mathfrak{I}_\mathfrak{s}} (\varpi - \vartheta)(\varpi - \vartheta - 1)\vartheta$

$\therefore\ \overline{M_1}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})) = \vartheta(\varpi - \vartheta)(\varpi - \vartheta - 1).$ □

Theorem 6.2. *Let $\mathfrak{R}$ be a ring of order $p^t q$. Then,*

$$\overline{M_2}(\mathfrak{G}_s(I)) = \overline{M_1}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})).\frac{\vartheta}{2}.$$

Proof. Let $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ be the S-prime graph. Then,

$$\begin{aligned}
\overline{M_2}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})) &= \textstyle\sum_{\mathfrak{x}\mathfrak{y} \notin \mathcal{E}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}). \\
&= \textstyle\sum_{\mathfrak{x},\mathfrak{y} \notin \mathfrak{I}_\mathfrak{s}} (\varpi - \vartheta)\dfrac{(\varpi - \vartheta - 1)}{2}\vartheta^2 \\
&= \dfrac{(\varpi - \vartheta)(\varpi - \vartheta - 1)}{2}.\vartheta^2 \\
&= \Big[\vartheta(\varpi - \vartheta)(\varpi - \vartheta - 1)\Big]\dfrac{\vartheta}{2}
\end{aligned}$$

Hence, $\overline{M_2}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})) = \overline{M_1}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})).\dfrac{\vartheta}{2}.$ □

Theorem 6.3. *Let $\mathfrak{R}$ be a ring of order pqr. Then,*

(i) $\overline{M_1}(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})) = 2(\varpi - 3)(\varpi - 4).$

(ii) $\overline{M_1}(\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})) = (\varpi - 3)\Big[3(\vartheta - 1) + \varpi\Big].$

(iii) $\overline{M_1}(\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})) = 2\Big[(\vartheta + 1)(\varpi - 4) + (\varpi - 1)(\vartheta + 2)\Big].$

Proof. (i) Let $\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})$ be the S-prime graph. Then,

$$\begin{aligned}
\overline{M_1}(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})) &= \textstyle\sum_{\mathfrak{x}\mathfrak{y} \notin \mathcal{E}(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s}))}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})] \\
&= \textstyle\sum_{\mathfrak{x},\mathfrak{y} \notin \mathfrak{I}_\mathfrak{s}}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})] \\
&= \Big[(\varpi - 3) + (\varpi - 3)\Big](\varpi - 4) \\
&= 2(\varpi - 3)(\varpi - 4).
\end{aligned}$$

(ii) Let $\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})$ be the S-prime graph. Consider the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ as $\downarrow pq$ and $\mathfrak{d}(\mathfrak{x}), \mathfrak{d}(\mathfrak{y})$ are defined in (5.2).

Then,

$$\begin{aligned}\overline{M_1}(\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})) &= \textstyle\sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}}))}[\mathfrak{d}(\mathfrak{x})+\mathfrak{d}(\mathfrak{y})] \\ &= \textstyle\sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}}[\mathfrak{d}(\mathfrak{x})+\mathfrak{d}(\mathfrak{y})] \\ &= \textstyle\sum_{\mathfrak{x}=r,\mathfrak{y}\neq r}[\mathfrak{d}(\mathfrak{x})+\mathfrak{d}(\mathfrak{y})] + \sum_{\mathfrak{x},\mathfrak{y}\neq r}[\mathfrak{d}(\mathfrak{x})+\mathfrak{d}(\mathfrak{y})] \\ &= (\varpi-3)\Big[\vartheta+(\varpi-3)\Big]+\vartheta\Big[(\varpi-3)+(\varpi-3)\Big] \\ &= (\varpi-3)\Big[3(\vartheta-1)+\varpi\Big].\end{aligned}$$

(iii) Let $\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})$ be the S-prime graph. Consider the S-prime ideal $\mathfrak{I}_{\mathfrak{s}}$ as $\downarrow pqr$ and $\mathfrak{d}(\mathfrak{x}),\mathfrak{d}(\mathfrak{y})$ are defined in (5.3). Then,

$$\begin{aligned}\overline{M_1}(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})) &= \textstyle\sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}}))}[\mathfrak{d}(\mathfrak{x})+\mathfrak{d}(\mathfrak{y})] \\ &= \textstyle\sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}}[\mathfrak{d}(\mathfrak{x})+\mathfrak{d}(\mathfrak{x})] \\ &= \textstyle\sum_{\mathfrak{x},\mathfrak{y}=\mathfrak{M}_{\mathfrak{k}}}[\mathfrak{d}(\mathfrak{x})+\mathfrak{d}(\mathfrak{y})] + \sum_{\mathfrak{x}=\mathfrak{M}_{\mathfrak{k}},\mathfrak{y}\neq\mathfrak{M}_{\mathfrak{k}}}[\mathfrak{d}(\mathfrak{x})+\mathfrak{d}(\mathfrak{y})] \\ &= 2(\vartheta+1)(\varpi-4)+(\varpi-1)\Big[\vartheta+1+(\vartheta+3)\Big] \\ &= 2(\vartheta+1)(\varpi-4)+(\varpi-1)(2\vartheta+4) \\ &= 2(\vartheta+1)(\varpi-4)+2(\varpi-1)(\vartheta+2) \\ &= 2\Big[(\vartheta+1)(\varpi-4)+(\varpi-1)(\vartheta+2)\Big].\end{aligned}$$

□

Theorem 6.4. *Let $\mathfrak{R}$ be a ring of order pqr. Then,*

(i) $\overline{M_2}(\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}})) = (\varpi-4)\vartheta^2$.

(ii) $\overline{M_2}(\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})) = 2\vartheta(\vartheta+2)^2$.

(iii) $\overline{M_2}(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})) = (\vartheta+1)\Big[(\vartheta+1)(\varpi-4)+(\vartheta+3)(\varpi-1)\Big]$.

Proof. (i) Let $\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}})$ be the S-prime graph. Then,

$$\begin{aligned}\overline{M_2}(\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}})) &= \textstyle\sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}}))}[\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})] \\ &= \textstyle\sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}}[\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})] \\ &= (\varpi-4)\vartheta\vartheta \\ &= (\varpi-4)\vartheta^2.\end{aligned}$$

(ii) Let $\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})$ be the S-prime graph. Consider the S-prime ideal $\mathfrak{I}_{\mathfrak{s}}$ as $\downarrow pq$ and $\mathfrak{d}(\mathfrak{x}),\mathfrak{d}(\mathfrak{y})$ are defined in (5.2). Then,

$$\overline{M_2}(\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})) = \textstyle\sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}}))}[\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})]$$

$$= \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}} [\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})]$$

$$= \sum_{\mathfrak{x}=r,\mathfrak{y}\neq r} [\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})] + \sum_{\mathfrak{x},\mathfrak{y}\neq r} [\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})]$$

$$= \vartheta(\vartheta+2)(\vartheta+2) + (\vartheta+2)(\vartheta+2)\vartheta$$

$$= 2\vartheta(\vartheta+2)^2.$$

(iii) Let $\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})$ be the S-prime graph. Consider the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ as $\downarrow pqr$ and $\mathfrak{d}(\mathfrak{x}), \mathfrak{d}(\mathfrak{y})$ are defined in (5.3). Then,

$$\overline{M_2}(\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s}))} [\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})]$$

$$= \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}} [\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})]$$

$$= \sum_{\mathfrak{x},\mathfrak{y}=\mathfrak{M}_\mathfrak{k}} [\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})] + \sum_{\mathfrak{x}=\mathfrak{M}_\mathfrak{k},\mathfrak{y}\neq\mathfrak{M}_\mathfrak{k}} [\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})]$$

$$= (\vartheta+1)(\vartheta+1)(\varpi-4) + (\vartheta+1)(\vartheta+3)(\varpi-1)$$

$$= (\vartheta+1)\Big[(\vartheta+1)(\varpi-4) + (\vartheta+3)(\varpi-1)\Big].$$ □

7 Randi ́c Index of S-prime Graph

In this section, $R(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))$ of $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ of the S-prime ideals $\mathfrak{I}_\mathfrak{s}$ are generalized.

Theorem 7.1. *Let $\mathfrak{R}$ be a ring of order p^tq. Then,*

$$R(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})) = \vartheta\Big[\frac{(\vartheta-1)}{2(\varpi-1)} + \frac{(\varpi-\vartheta)}{\sqrt{\vartheta(\varpi-1)}}\Big].$$

Proof. Let $\mathfrak{x}\mathfrak{y} \in \mathcal{E}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))$ be the edge set of the S-prime graph and the degrees of the vertices are defined in (5.1). Then,

$$R(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))} \frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}}$$

$$= \sum_{\mathfrak{x},\mathfrak{y}\in\mathfrak{I}_\mathfrak{s}} \frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} + \sum_{\mathfrak{x}\in\mathfrak{I}_\mathfrak{s},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}} \frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}}$$

$$= \frac{\vartheta(\vartheta-1)}{2\sqrt{(\varpi-1)^2}} + \frac{\vartheta(\varpi-\vartheta)}{\sqrt{\vartheta(\varpi-1)}}$$

Thus, $R(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})) = \vartheta\Big[\frac{(\vartheta-1)}{2(\varpi-1)} + \frac{(\varpi-\vartheta)}{\sqrt{\vartheta(\varpi-1)}}\Big].$ □

Theorem 7.2. *Let $\mathfrak{R}$ be a ring of order pqr. Then,*

(i) $R(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})) = 1 + \frac{(\varpi-3)(\varpi-4)}{\sqrt{\vartheta(\varpi-1)}}$

(ii) $R(\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})) = 1 + \dfrac{1}{(\varpi-1)} + \dfrac{\vartheta}{\sqrt{\vartheta(\varpi-1)}} + \dfrac{8}{\sqrt{(\varpi-1)(\vartheta+2)}}.$

(iii) $R(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})) = 3\Big[\dfrac{1}{\sqrt{(\varpi-1)(\vartheta+1)}} + \dfrac{1}{\sqrt{(\varpi-1)(\vartheta+3)}} + \dfrac{1}{\sqrt{(\vartheta+1)(\vartheta+3)}} + \dfrac{1}{(\vartheta+3)}\Big].$

Proof. (i) Let $\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}})$ be the S-prime graph. Then,

$$\begin{aligned} R(\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}})) &= \textstyle\sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}(\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}}))} \dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} \\ &= \textstyle\sum_{\mathfrak{x},\mathfrak{y}\in\mathfrak{I}_{\mathfrak{s}}} \dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} + \sum_{\mathfrak{x}\in\mathfrak{I}_{\mathfrak{s}},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} \dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} \\ &= \dfrac{(\varpi-1)}{\sqrt{(\varpi-1)(\varpi-1)}} + \dfrac{(\varpi-3)(\varpi-4)}{\sqrt{(\varpi-1)\vartheta}} \\ &= 1 + \dfrac{(\varpi-3)(\varpi-4)}{\sqrt{\vartheta(\varpi-1)}}. \end{aligned}$$

(ii) Let $\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})$ be the S-prime graph. Consider the S-prime ideal $\mathfrak{I}_{\mathfrak{s}}$ as $\downarrow pq$. Let $\mathfrak{x}\mathfrak{y} \in \mathcal{E}(\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}}))$ be the edge set of $\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})$ and the degrees of the vertices are defined in (5.2). Then,

$$\begin{aligned} R(\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})) &= \textstyle\sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}(\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}}))} \dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} \\ &= \textstyle\sum_{\mathfrak{x},\mathfrak{y}\in\mathfrak{I}_{\mathfrak{s}}} \dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} + \sum_{\mathfrak{x}\in\mathfrak{I}_{\mathfrak{s}},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} \dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} + \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} \dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} \\ &= 1 + \dfrac{1}{(\varpi-1)} + \dfrac{\vartheta}{\sqrt{\vartheta(\varpi-1)}} + \dfrac{8}{\sqrt{(\varpi-1)(\vartheta+2)}}. \end{aligned}$$

(iii) Let $\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})$ be the S-prime graph. Consider the S-prime ideal $\mathfrak{I}_{\mathfrak{s}}$ as $\downarrow pqr$. Let $\mathfrak{x}\mathfrak{y} \in \mathcal{E}(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}}))$ be the edge set of $\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})$ and the degrees of the vertices are defined in (5.3). Then,

$$\begin{aligned} R(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})) &= \textstyle\sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}}))} \dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} \\ &= \textstyle\sum_{\mathfrak{x}\in\mathfrak{I}_{\mathfrak{s}},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} \dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} + \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} \dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} \\ &= \Big[\dfrac{3}{\sqrt{(\varpi-1)(\vartheta+1)}} + \dfrac{3}{\sqrt{(\varpi-1)(\vartheta+3)}}\Big] + \Big[\dfrac{3}{\sqrt{(\vartheta+1)(\vartheta+3)}} + \dfrac{3}{\sqrt{(\vartheta+3)^2}}\Big] \\ &= 3\Big[\dfrac{1}{\sqrt{(\varpi-1)(\vartheta+1)}} + \dfrac{1}{\sqrt{(\varpi-1)(\vartheta+3)}} + \dfrac{1}{\sqrt{(\vartheta+1)(\vartheta+3)}} + \dfrac{1}{(\vartheta+3)}\Big]. \end{aligned}$$

□

8 Conclusion

In this paper, the meet subset and a new ideal called the S-prime ideal in a lattice and S-meet semilattice are defined and it is shown that the prime ideal of a lattice is also an S-prime ideal of a lattice $\mathfrak{L}$. Then, a new graph for the S-meet semilattice $(L_s, \wedge, \subseteq)$ is introduced and explained with necessary examples and their degree based topological measures are generalized for the S-prime graph of S-meet semilattice L_s.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of this manuscript.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Gratzer G, Schmidt ET. Standard ideal in lattices. Acta math. Acad. Sci. Hung. 1961;12:17-86.

[2] Noor ASA, Latif MA. Standard n-ideals of a lattice. Southeast Asian Bull. Math. 1997;4:185-192.

[3] Latif MA, Noor ASA. n-Ideals of a Lattice. The Rajshahi University Studies, Part-B.1994;22:173-180.

[4] Noor ASA, Latif MA. Finitely generated n-ideals of a lattice. Southeast Asian Bull. Math. 1998;4:73-79.

[5] Noor ASA, Latif MA. Properties of Standard n-ideals of a lattice. Southeast Asian Bull. Math. 2000;24:403-409.

[6] Meenakshi P. Wasadikar, Karuna T. Gaikwad. On 2-absorbing and weakly 2-absorbing ideals of lattices. Mathematical Sciences International Research Journal. 2015;4(2):82 -85.

[7] Badawi A. On 2-absorbing ideals of commutative rings. Bull. Austral. Math. Soc. 2007;75:417-429.

[8] Badawi A, Darani AY. On weakly 2-absorbing ideals of commutative rings. Houston J. Math. 2013;39(2):441-452.

[9] Ali Akbar Estaji, Toktam Haghdadi. On n-absorbing Ideals in a Lattice. Kragujevac Journal of Mathematics. 2021;45(4):597-605.

[10] Anderson DF, Ayman Badawi. On n-absorbing ideals of commutative rings. 2011;39:1646-1672.

[11] Ali A, Hafizur Rahman RM, Noor ASA, Mizanur Rahman M. On semiprime n-ideals of lattices. Annals of Pure and Applied Mathematics. 2012;21(1):10-17.

[12] Ayub Ali M, Noor ASA. Some properties of modular n-ideals of a lattice. Journal of Physical Sciences. 2010;14:173-180.

[13] Ahmed Hamed, Achraf Malek. S-prime ideals of a commutative ring. Beitrage zur Algebra and Geometrie. 2020;61:533-542.

[14] Kalamani D, Mythily CV. S-prime ideal graph of a finite commutative ring. Advances and Applications in Mathematical Sciences. 2023;22:861-872.

[15] Mythily CV, Kalamani D. Study on S-prime Ideal as Nilpotent Ideal. Journal of Applied Mathematics and Informatics; 2024 (In press).

[16] Kalamani D, Mythily CV. Some new results on S-prime ideals of a finite commutative ring as S-meet semilattice. Applied and Computational Mathematics. 2024;13 (4):105-110.

[17] Ashrafi AR, Doslic T, Hamzeh A. The Zagreb coindices of graph operations. Discrete Applied Mathematics. 2010;158;1571-1578.

[18] Khalifeh MH, Yousefi-Azari H, Ashrafi AR. The first and second Zagreb indices of some graph operations. Discrete Applied Mathematics. 2009;157:804-811.

[19] Kiruthika G, Kalamani D. Degree based partition of the power graph of the finite Abelian group. Malaya Journal of Matematik. 2020;1:66-71.

[20] Gratzer G. General lattice theory. Academic Press. New York San Francisco; 1978.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
https://www.sdiarticle5.com/review-history/122295