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Procedure for Exact Solutions of Stiff Ordinary Differential Equations Systems

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Abstract

In this work, we present a technique for the analytical solution of systems of stiff ordinary differential equations (SODEs) using the power series method (PSM). Three SODEs systems are solved to show that PSM can find analytical solutions of SODEs systems in convergent series form. Additionally, we propose a post-treatment of the power series solutions with the Laplace-Padé (LP) resummation method as a powerful technique to find exact solutions. The proposed method gives a simple procedure based on a few straightforward steps.

Keywords: Stiff ordinary differential equations, Power series method, Laplace transform, Padé approximant, Analytical solutions.

2010 Mathematics Subject Classification: 34L30

1 Introduction

Systems of stiff ordinary differential equations (SODEs) arise in modelling different physical problems with widely differing time scales; therefore, they have been extensively studied for more than two decades [1]. This kind of equations arise in fluid mechanics, elasticity, electrical networks, chemical reactions, vibrations, weather prediction, biology, among others [2], [3]. A stiff problem can exhibit several periods of rapid change; this can be even more complicated because some of the variables may change rapidly and the others slow. Therefore, explicit numerical methods like classic explicit Euler or fourth-order Runge-Kutta schemes will require a very small time step to accurately follow the solution of SODEs, resulting in expensive computation [4]. Implicit numerical A-stable methods with variable step size [5], [6], [7], [8], [9], [10] can solve some specific stiff problems efficiently, however, there is not a standard method to numerically solve SODEs. In addition, during recent years exponential integrators gained attention for application to solve large stiff problems [11]. In general, stiffness is an efficiency issue [3]. What is more, due to a sort of discretization, explicit, implicit, or exponential integrator methods are affected by round-off errors, which sooner or later can lead to wrong or inaccurate results [12], [13].

In order to circumvent the aforementioned issues, some semi-analytical approaches have been proposed to solve SODEs as: homotopy perturbation method (HPM) [12], [14], Laplace transform homotopy perturbation [2], Adomian decomposition method (ADM) [15], variational iteration method (VIM) [13], among others. Unfortunately, such methods are usually complicated to apply or require initial approximations to succeed. Therefore, in this work, we present the application of a hybrid technique combining PSM [16], [17], [18], [19], [20], Laplace Transform (LT) and Padé Approxi-mant (PA) [21] to find analytical solutions for SODEs [22], [23], [24], [25], [26], [27], [28], [29]. Using the PSM, we first find the solutions to SODEs in convergent series form. The truncated PSM's series solution is improved by applying LT to it and converting the resulting series into a meromorphic function by forming its PA. Finally, to obtain the analytical solution, we apply the inverse LT to the PA. This hybrid method which we call (LPPSM) combines the PSM with Laplace-Padé resummation. It greatly improves the PSM's truncated series solutions in convergence rate. The Laplace-Padé resummation method in fact enlarges the convergence domain of the truncated power series and usually leads to the exact solution.

It is worth mentioning that no secular terms (noise terms) are generated by the proposed method as the homotopy perturbation based techniques. This property of the PSM improves the efficiency of the LPPSM in comparison to the perturbation based methods. Furthermore, the LPPSM does not require any perturbation parameter as the perturbation based techniques. Finally, the LPPSM can be coded using computer algebra software like Mathematica or Maple.

The rest of this paper is organized as follows. The next section is devoted to the basic concept of the PSM. In section 3 we discuss the main idea behind the Padé approximant. The concept of the Laplace-Padé resummation method is given in section 4. The application of PSM to SODEs systems is depicted in section 5. In section 6, we apply LPPSM to find the analytical solutions of three linear and nonlinear SODEs systems. In section 7, we give a brief discussion. Finally, we draw a conclusion in the last section.

2 Basic Concept of Power Series Method

Consider the nonlinear differential equation

$$A(u) - f(t) = 0, \qquad t \in \Omega, \tag{1}$$

with the boundary conditions

$$B\left(u, \partial u/\partial \eta\right) = 0, \qquad t \in \Gamma, \tag{2}$$

where the operator A is a general differential operator, $f(t)$ is a known analytic function, B is a boundary operator, and Γ is the boundary of Ω.

According to the PSM [30, 31], the solution of the differential equation (1) is assumed to have the form

$$u\left(t\right) = \sum_{n=0}^{\infty} u_n t^n, \tag{3}$$

where the unknowns coefficients $u_0, u_1, \ldots$ are to be determined by the PSM.

The procedure for the PSM can be summarized as follows:

1. We substitute (3) into (1), then regroup the terms according to the powers of t.

2. We equate all coefficients of powers of t to zero in the resulting polynomial.

3. The boundary conditions (2) are substituted into (3) to generate an algebraic equation for each boundary condition.

4. Aforementioned steps generate algebraic linear equations for the unknowns of (3).

5. Finally, we solve the resulting algebraic linear equations to find the coefficients $u_0, u_1, \ldots$

3 Padé Approximant

Given an analytical function $u(t)$ with Maclaurin's expansion

$$u\left(t\right)=\sum_{n=0}^{\infty}u_nt^n,\ 0\leq t\leq T. \tag{4}$$

The Padé approximant to $u\left(t\right)$ of order $[L, M]$ which we denote by $[L/M]_u\left(t\right)$ is defined by [21]

$$\left[L/M\right]_u\left(t\right)=\frac{p_0+p_1t+\ldots+p_Lt^L}{1+q_1t+\ldots+q_Mt^M}, \tag{5}$$

where we considered $q_0 = 1$, and the numerator and denominator have no common factors.

The numerator and the denominator in (5) are constructed so that $u\left(t\right)$ and $[L/M]_u\left(t\right)$ and their derivatives agree at $t = 0$ up to $L + M$. That is

$$u(t)-\left[L/M\right]_u\left(t\right)=O\left(t^{L+M+1}\right). \tag{6}$$

From (6), we have

$$u\left(t\right)\sum_{n=0}^{M}q_nt^n-\sum_{n=0}^{L}p_nt^n=O\left(t^{L+M+1}\right). \tag{7}$$

From (7), we get the following algebraic linear systems

$$\begin{cases} u_Lq_1+\ldots+u_{L-M+1}q_M=-u_{L+1} \\ u_{L+1}q_1+\ldots+u_{L-M+2}q_M=-u_{L+2} \\ \vdots \\ u_{L+M-1}q_1+\ldots+u_Lq_M=-u_{L+M}, \end{cases} \tag{8}$$

and

$$\begin{cases} p_0 = u_0 \\ p_1 = u_1 + u_0 q_1 \\ \vdots \\ p_L = u_L + u_{L-1} q_1 + \ldots + u_0 q_L. \end{cases} \tag{9}$$

From (8), we calculate first all the coefficients q_n, $1 \leq n \leq M$. Then, we determine the coefficients p_n, $0 \leq n \leq L$ from (9).

Here, we point out that for a fixed value of $L + M + 1$, the error (6) is minimum when the degree of the numerator is equal to that of the denominator of (5) or when the numerator is one degree higher than the denominator.

4 Laplace-Padé Resummation Method

Several approximate analytical methods lead to solutions in the form of power series (polynomial). Nevertheless, in some cases, this type of solutions do not have large domains of convergence. Therefore, to enlarge the domain of convergence of solutions or find the exact solutions, Laplace-Padé resummation method is often used [22], [23], [24], [25], [26], [27], [28], [29].

The Laplace-Padé method can be summarized as follows:

1. First, we apply Laplace transform to power series (3).

2. Next, we substitute s by $1/t$ in the resulting equation.

3. Then, we convert the transformed series into a meromorphic function by forming its Padé approximant of order $[N/M]$. Here, N and M are chosen arbitrarily with values smaller than the order of the power series. In this step, the Padé approximant extends the domain of convergence of the truncated series solution to obtain a better accuracy.

4. After that, we substitute t by $1/s$.

5. Finally, by applying the inverse Laplace s transform, we get the approximate or exact solution.

   We point out here that Laplace-Padé resummation method is useful when the inverse Laplace s transform to the Padé approximant in step 5 can be found.

5 Application of PSM to Solve Stiff ODE Systems

Many real application problems in science and engineering often lead to the solution of stiff ODEs systems of the form

$$u^{'}(t) = f(t, u(t)),\ 0 \le t \le T, \tag{10}$$

$$u(0) = \eta. \tag{11}$$

We assume that the solution to initial value problem (10)-(11) exists, is unique and analytic. To simplify the exposition of the PSM, we integrate equation (10) once with respect to t and use the initial conditions (11) to obtain

$$u(t) = \eta + \int_0^t f(\tau, u(\tau))\, d\tau. \tag{12}$$

We note here that the time integration of equation (10) is not relevant to the solution procedure. Thus, one can apply the PSM directly to equation (10).

In view of the PSM, the solution $u(t)$ of (10)-(11) is assumed to have the form

$$u(t) = u_0 + u_1 t + u_2 t^2 + \ldots, \tag{13}$$

where u_n, $n = 0, 1, 2, \ldots$ are unknown coefficients to be determined later by the PSM.

To find the solution of (10)-(11), we substitute expansion (13) into equation (12) and equate the coefficients of powers of t to zero in the resulting equation to obtain recursions for the coefficients u_n, $n = 0, 1, 2, \ldots$ Finally, we use expansion (13) to derive the exact solution as power series.

The convergence region for the solutions series obtained may be small. Thus, we propose to apply the Laplace-Padé post-treatment to PSM's truncated series (which we call LPPSM) to enlarge the convergence region as depicted in the next section.

6 Cases Study

In this section, we shall demonstrate through three examples that the LPPSM is effective and accurate in solving stiff systems of ordinary differential equations.

6.1 Stiff nonlinear system of ordinary differential equations

Consider the following stiff nonlinear system of ordinary differential equations of two variables [8]

$$u' = -1002u + 1000v^2, \tag{14}$$

$$v' = u - v - v^2,\ t \ge 0, \tag{15}$$

with the initial conditions

$$u(0) = 1, \quad v(0) = 1. \tag{16}$$

The exact analytical solution of initial value problem (14)-(16) is

$$u(t) = e^{-2t}, \quad v(t) = e^{-t}. \tag{17}$$

In order to simplify the exposition of the LPPSM presented in sections 4 and 5 to solve (14)-(16), we first integrate equations (14)-(15) once with respect to t and use the initial conditions (16) to get

$$u(t) - 1 + \int_0^t 1002u(\tau) - 1000v^2(\tau)\, d\tau \quad = \quad 0, \tag{18}$$

$$v(t) - 1 - \int_0^t u(\tau) - v(\tau) - v^2(\tau)\, d\tau \quad = \quad 0. \tag{19}$$

In view of the PSM, we assume that the solution components $u(t)$ and $v(t)$ have the form

$$u(t) = u_0 + u_1 t + u_2 t^2 + \ldots, \tag{20}$$

and

$$v(t) = v_0 + v_1 t + v_2 t^2 + \ldots, \tag{21}$$

respectively, where u_n and v_n, $n = 0, 1, 2, \ldots$ are unknown coefficients to be determined later by the PSM.

Then, we substitute (20) and (21) into (18)-(19) to obtain

$$\sum_{n=0}^{\infty} u_n t^n - 1 + \int_0^t 1002 \sum_{n=0}^{\infty} u_n \tau^n - 1000 \left(\sum_{n=0}^{\infty} v_n \tau^n \right)^2 d\tau \quad = \quad 0, \tag{22}$$

$$\sum_{n=0}^{\infty} v_n t^n - 1 - \int_0^t \sum_{n=0}^{\infty} u_n \tau^n - \sum_{n=0}^{\infty} v_n \tau^n - \left(\sum_{n=0}^{\infty} v_n \tau^n \right)^2 d\tau \quad = \quad 0. \tag{23}$$

This yields

$$(u_0 - 1) + \sum_{n=1}^{\infty} \left(u_n + (1002/n)\, u_{n-1} - (1000/n) \sum_{k=0}^{n-1} v_k v_{n-1-k} \right) t^n = 0, \tag{24}$$

$$(v_0 - 1) + \sum_{n=1}^{\infty} \left(v_n - (1/n) u_{n-1} + (1/n) v_{n-1} + (1/n) \sum_{k=0}^{n-1} v_k v_{n-1-k} \right) t^n = 0. \tag{25}$$

Equating all coefficients of powers of t to zero in (24) and (25), we have

$$u_0 = 1,\ v_0 = 1,$$

and the following recursions for the unknown coefficients u_n and v_n

$$u_n = -\frac{1}{n}\left(1002u_{n-1} - 1000\sum_{k=0}^{n-1} v_k v_{n-1-k}\right), \tag{26}$$

$$v_n = \frac{1}{n}\left(u_{n-1} - v_{n-1} - \sum_{k=0}^{n-1} v_k v_{n-1-k}\right),\ n = 1, 2, 3, \ldots \tag{27}$$

From these recursions, we compute some coefficients

$$\begin{aligned}
u_1 &= -2, \quad v_1 = -1,\\
u_2 &= 2, \quad v_2 = \frac{1}{2},\\
u_3 &= -\frac{4}{3}, \quad v_3 = -\frac{1}{6},\\
u_4 &= \frac{2}{3}, \quad v_4 = \frac{1}{24},\\
u_5 &= -\frac{4}{15}, \quad v_5 = -\frac{1}{120},\\
&\ldots
\end{aligned} \tag{28}$$

Then using equations (20)-(21) and the coefficients above, we obtain

$$u(t) = 1 - 2t + 2t^2 - \frac{4t^3}{3} + \frac{2t^4}{3} - \frac{4t^5}{15}, \tag{29}$$

$$v(t) = 1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} - \frac{t^5}{120}. \tag{30}$$

To increase the accuracy of this approximate solution, we apply Laplace-Pad é post-treatment to it. First, t-Laplace transform is applied to (29) and (30). Second, s is substituted by $1/t$ then t-Padé approximant is applied to the transformed series. Finally, t is substituted by $1/s$ and the inverse Laplace s-transform is applied to the resulting expressions to find the approximate solution or exact solution.

Applying Laplace transform to (29) and (30) yields

$$\mathcal{L}[u(t)] = \frac{1}{s} - \frac{2}{s^2} + \frac{4}{s^3} - \frac{8}{s^4} + \frac{16}{s^5} - \frac{32}{s^6}, \tag{31}$$

and

$$\mathcal{L}[v(t)] = \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s^4} + \frac{1}{s^5} - \frac{1}{s^6}. \tag{32}$$

For the sake of simplicity we let $s = 1/t$, then

$$\mathcal{L}\left[u\left(t\right)\right] = t - 2t^2 + 4t^3 - 8t^4 + 16t^5 - 32t^6, \tag{33}$$

and

$$\mathcal{L}\left[v\left(t\right)\right] = t - t^2 + t^3 - t^4 + t^5 - t^6. \tag{34}$$

All of the $[L/M]$ t-Padé approximants of (33) and (34) with $L \geq 1$ and $M \geq 1$ and $L + M \leq 6$ yield

$$[L/M]_u = \frac{t}{1+2t}, \tag{35}$$

and

$$[L/M]_v = \frac{t}{1+t}. \tag{36}$$

Now since $t = 1/s$, we obtain $[L/M]_u$ and $[L/M]_v$ in terms of s as follows

$$[L/M]_u = \frac{1}{2+s}, \tag{37}$$

and

$$[L/M]_v = \frac{1}{1+s}. \tag{38}$$

Finally, applying the inverse LT to the Padé approximants (37) and (38), we obtain the approximate solution which is in this case the exact solution (17). Note that if we take more terms in series (29) and (30), we get the same exact solution.

6.2 Strongly stiff linear system of ordinary differential equations

Consider the following stiff linear system of ordinary differential equations of two variables [8]

$$u' = -500000.5u + 499999.5v, \tag{39}$$

$$v' = 499999.5u - 500000.5v,\ t \geq 0, \tag{40}$$

with the initial conditions

$$u\left(0\right) = 0, \quad v\left(0\right) = 2. \tag{41}$$

System (39)-(40) is strongly stiff with a stiffness ratio of 10^6. The exact analytical solution of initial value problem (39)-(41) is

$$u\left(t\right) = e^{-t} - e^{-10^6 t}, \quad v\left(t\right) = e^{-t} + e^{-10^6 t}. \tag{42}$$

In order to simplify the exposition of the LPPSM presented in sections 4 and 5 to solve (39)-(41), we first integrate equations (39)-(40) once with respect to t and use the initial conditions (41) to get

$$u(t) + \int_0^t 500000.5u(\tau) - 499999.5v(\tau)\, d\tau = 0, \quad (43)$$

$$v(t) - 2 - \int_0^t 499999.5u(\tau) - 500000.5v(\tau)\, d\tau = 0. \quad (44)$$

In view of the PSM, we assume that the solution components $u(t)$ and $v(t)$ have the form

$$u(t) = u_0 + u_1 t + u_2 t^2 + \ldots, \quad (45)$$

and

$$v(t) = v_0 + v_1 t + v_2 t^2 + \ldots, \quad (46)$$

respectively, where u_n and v_n, $n = 0, 1, 2, \ldots$ are unknown coefficients to be determined later by the PSM.

Then, we substitute (45) and (46) into (43)-(44) to have

$$\sum_{n=0}^{\infty} u_n t^n + \int_0^t \sum_{n=0}^{\infty} \left(500000.5u_n - 499999.5v_n \right) \tau^n d\tau = 0, \quad (47)$$

$$\sum_{n=0}^{\infty} v_n t^n - 2 - \int_0^t \sum_{n=0}^{\infty} \left(499999.5u_n - 500000.5v_n \right) \tau^n d\tau = 0. \quad (48)$$

This yields

$$u_0 + \sum_{n=1}^{\infty} \left(u_n + (500000.5/n)u_{n-1} - (499999.5/n)\, v_{n-1} \right) t^n = 0, \quad (49)$$

$$(v_0 - 2) + \sum_{n=1}^{\infty} \left(v_n - (499999.5/n)\, u_{n-1} + (500000.5/n)v_{n-1} \right) t^n = 0. \quad (50)$$

Equating all coefficients of powers of t to zero in (49) and (50), we have

$$u_0 = 0, \quad v_0 = 2,$$

and the following recursion for the unknown coefficients u_n and v_n

$$u_n = -\frac{1}{n}\left(500000.5u_{n-1} - 499999.5v_{n-1} \right), \quad (51)$$

$$v_n = \frac{1}{n}\left(499999.5u_{n-1} - 500000.5v_{n-1} \right),\ n = 1, 2, 3, \ldots \quad (52)$$

From these recursions, we compute some coefficients

$$\begin{aligned} u_1 &= -1+10^6, \quad v_1 = -1-10^6, \\ u_2 &= -\frac{10^{12}-1}{2}, \quad v_2 = \frac{10^{12}+1}{2}, \\ u_3 &= \frac{10^{18}-1}{6}, \quad v_3 = -\frac{10^{18}+1}{6}, \\ u_4 &= -\frac{10^{24}-1}{24}, \quad v_4 = \frac{10^{24}+1}{24}, \\ u_5 &= \frac{10^{30}-1}{120}, \quad v_5 = -\frac{10^{30}+1}{120}, \\ &\cdots \end{aligned} \tag{53}$$

Then using (45)-(46) and the coefficients above, we obtain

$$u(t) = \left(-1+10^6\right)t - \frac{10^{12}-1}{2}t^2 + \frac{10^{18}-1}{6}t^3 - \frac{10^{24}-1}{24}t^4 + \frac{10^{30}-1}{120}t^5, \tag{54}$$

$$v(t) = 2 - \left(1+10^6\right)t + \frac{10^{12}+1}{2}t^2 - \frac{10^{18}+1}{6}t^3 + \frac{10^{24}+1}{24}t^4 - \frac{10^{30}+1}{120}t^5. \tag{55}$$

To increase the accuracy of this approximate solution, we apply Laplace-Pad é post-treatment to it. First, t-Laplace transform is applied to (54) and (55). Second, s is substituted by $1/t$ then t-Padé approximant is applied to the transformed series. Finally, t is substituted by $1/s$ and the inverse Laplace s-transform is applied to the resulting expressions to find the approximate solution or exact solution.

Applying Laplace transform to (54) and (55) yields

$$\mathcal{L}[u(t)] = 999999s^{-2} - 999999999999s^{-3} + 999999999999999999s^{-4}, \tag{56}$$

and

$$\mathcal{L}[v(t)] = 2s^{-1} - 1000001s^{-2} + 1000000000001s^{-3} - 1000000000000000001s^{-4}. \tag{57}$$

For the sake of simplicity we let $s = 1/t$, then

$$\mathcal{L}[u(t)] = 999999t^2 - 999999999999t^3 + 999999999999999999t^4, \tag{58}$$

and

$$\mathcal{L}[v(t)] = 2t - 1000001t^2 + 1000000000001t^3 - 1000000000000000001t^4. \tag{59}$$

All of the $[L/M]$ t-Padé approximants of (58) and (59) with $L \geq 1$ and $M \geq 1$ and $L + M \leq 4$ yield

$$[L/M]_u = \frac{999999t^2}{1000000t^2 + 1000001t + 1}, \tag{60}$$

and

$$[L/M]_v = \frac{1000001t^2 + 2t}{1000000t^2 + 1000001t + 1}. \tag{61}$$

Now since $t = 1/s$, we obtain $[L/M]_u$ and $[L/M]_v$ in terms of s as follows

$$[L/M]_u = \frac{999999}{1000000 + 1000001s + s^2}, \tag{62}$$

and

$$[L/M]_v = \frac{1000001 + 2s}{1000000 + 1000001s + s^2}. \tag{63}$$

Finally, applying the inverse LT to the Padé approximants (62) and (63), we obtain the approximate solution which is in this case the exact solution (42).

6.3 Stiff linear system of ordinary differential equations

Consider the following stiff linear system of ordinary differential equations of three variables [32]

$$\begin{aligned} u' + 0.1u + 49.9v &= 0, \\ v' + 50v &= 0, \\ w' - 70v + 120w &= 0,\ t \geq 0, \end{aligned} \tag{64}$$

with the initial conditions

$$u(0) = 2, \quad v(0) = 1, \quad w(0) = 2. \tag{65}$$

System (64) is stiff with a stiffness ratio of 1200. The exact analytical solution of initial value problem (64)-(65) is

$$u(t) = e^{-0.1t} + e^{-50t}, \quad v(t) = e^{-50t}, \quad w(t) = e^{-50t} + e^{-120t}. \tag{66}$$

In order to simplify the exposition of the LPPSM presented in sections 4 and 5 to solve (64)-(65), we first integrate equations (64) once with respect to t and use the initial conditions (65) to get

$$\begin{aligned} u(t) - 2 + \int_0^t 0.1u(\tau) + 49.9v(\tau)\, d\tau &= 0, \\ v(t) - 1 + 50\int_0^t v(\tau)\, d\tau &= 0, \\ w(t) - 2 - \int_0^t 70v(\tau) - 120w(\tau)\, d\tau &= 0. \end{aligned} \tag{67}$$

In view of the PSM, we assume that the solution components $u(t)$ and $v(t)$ have the form

$$\begin{aligned} u(t) &= u_0 + u_1 t + u_2 t^2 + \dots, \\ v(t) &= v_0 + v_1 t + v_2 t^2 + \dots, \\ w(t) &= w_0 + w_1 t + w_2 t^2 + \dots, \end{aligned} \tag{68}$$

where u_n,v_n and w_n, $n = 0, 1, 2, \dots$ are unknown coefficients to be determined later by the PSM.

Then, we substitute (68) into (67) to obtain

$$\begin{aligned} \sum_{n=0}^{\infty} u_n t^n - 2 + \int_0^t \sum_{n=0}^{\infty} (0.1u_n + 49.9v_n)\tau^n d\tau &= 0, \\ \sum_{n=0}^{\infty} v_n t^n - 1 + 50 \int_0^t \sum_{n=0}^{\infty} v_n \tau^n d\tau &= 0, \\ \sum_{n=0}^{\infty} w_n t^n - 2 - \int_0^t \sum_{n=0}^{\infty} (70v_n - 120w_n)\tau^n d\tau &= 0. \end{aligned} \tag{69}$$

This yields

$$\begin{aligned} (u_0 - 2) + \sum_{n=1}^{\infty} (u_n + (1/n)(0.1u_{n-1} + 49.9v_{n-1}))t^n &= 0, \\ (v_0 - 1) + \sum_{n=1}^{\infty} (v_n + (50/n)\, v_{n-1})\, t^n &= 0, \\ (w_0 - 2) + \sum_{n=1}^{\infty} (w_n + (1/n)\,(-70v_{n-1} + 120w_{n-1}))t^n &= 0. \end{aligned} \tag{70}$$

Equating all coefficients of powers of t to zero in system (70), we have

$$u_0 = 2,\ v_0 = 1,\ w_0 = 2,$$

and the following recursions for the unknown coefficients u_n, v_n and w_n

$$\begin{aligned} u_n &= -(1/n)\,(0.1u_{n-1} + 49.9v_{n-1}), \\ v_n &= -(50/n)\, v_{n-1}, \\ w_n &= (1/n)\,(70v_{n-1} - 120w_{n-1}),\ n = 1, 2, 3, \dots \end{aligned} \tag{71}$$

From these recursions, we compute some coefficients

$$
\begin{aligned}
u_1 &= -50.1, \quad v_1 = -50, \quad w_1 = -170, \\
u_2 &= 1250.005, \quad v_2 = 1250, \quad w_2 = 8450, \\
u_3 &= -20833.3335, \quad v_3 = -\frac{62500}{3}, \quad w_3 = -\frac{926500}{3}, \\
u_4 &= 2.60416666671 \times 10^5, \quad v_4 = \frac{781250}{3}, \quad w_4 = \frac{26701250}{3}, \\
& \dots
\end{aligned} \tag{72}
$$

Then using equations (68) and the coefficients above, we obtain

$$
\begin{aligned}
u(t) &= 2 - 50.1t + 1250.005t^2 - 20833.3335t^3 + 2.60416666671 \times 10^5 t^4, \\
v(t) &= 1 - 50t + 1250t^2 - \frac{62500}{3}t^3 + \frac{781250}{3}t^4, \\
w(t) &= 2 - 170t + 8450t^2 - \frac{926500}{3}t^3 + \frac{26701250}{3}t^4.
\end{aligned} \tag{73}
$$

To increase the accuracy of this approximate solution, we apply Laplace-Pad é post-treatment to it. First, t-Laplace transform is applied to (73). Second, s is substituted by $1/t$ then t-Padé approximant is applied to the transformed series. Finally, t is substituted by $1/s$ and the inverse Laplace s-transform is applied to the resulting expressions to find the approximate solution or exact solution.

Applying Laplace transform to (73) yields

$$
\begin{aligned}
\mathcal{L}[u(t)] &= 2s^{-1} - 50.1s^{-2} + 2500.01s^{-3} - 1.25000001 \times 10^5 s^{-4} + 6.25 \times 10^6 s^{-5}, \\
\mathcal{L}[v(t)] &= s^{-1} - 50s^{-2} + 2500s^{-3} - 125000s^{-4} + 6250000s^{-5}, \\
\mathcal{L}[w(t)] &= 2s^{-1} - 170s^{-2} + 16900s^{-3} - 1853000s^{-4} + 213610000s^{-5}.
\end{aligned} \tag{74}
$$

For the sake of simplicity we let $s = 1/t$, then

$$
\begin{aligned}
\mathcal{L}[u(t)] &= 2t - 50.1t^2 + 2500.01t^3 - 1.25000001 \times 10^5 t^4 + 6.25 \times 10^6 t^5, \\
\mathcal{L}[v(t)] &= t - 50t^2 + 2500t^3 - 125000t^4 + 6250000t^5, \\
\mathcal{L}[w(t)] &= 2t - 170t^2 + 16900t^3 - 1853000t^4 + 213610000t^5.
\end{aligned} \tag{75}
$$

All of the $[L/M]$ t-Padé approximants of (75) with $L \geq 1$ and $M \geq 1$ and $L+M \leq 5$ yield

$$
\begin{aligned}
[L/M]_u &= \frac{2t + 50.1t^2}{1 + 50.1t + 5t^2}, \\
[L/M]_v &= \frac{t}{1 + 50t}, \\
[L/M]_w &= \frac{2t + 170t^2}{1 + 170t + 6000t^2}.
\end{aligned} \tag{76}
$$

Now since $t = 1/s$, we obtain $[L/M]_u$, $[L/M]_v$ and $[L/M]_w$ in terms of s as follows

$$\begin{aligned} [L/M]_u &= \frac{2s + 50.1}{s^2 + 50.1s + 5}, \\ [L/M]_v &= \frac{1}{50 + s}, \\ [L/M]_w &= \frac{2s + 170}{s^2 + 170s + 6000}. \end{aligned} \tag{77}$$

Finally, applying the inverse LT to the Padé approximants (77), we obtain the approximate solution which is in this case the exact solution (66).

7 Discussion

This paper presents a hybrid method (LPPSM) combining the power series method (PSM) with its Laplace-Padé (LP) resummation as a useful analytical technique to solve stiff systems of ordinary differential equations (SODEs). We solved three linear and nonlinear SODEs systems by this technique and obtained the exact solutions. For each of the problems solved here, the PSM transformed the SODEs system into an easily solvable linear algebraic system for the coefficients of the power series solution. To improve the accuracy of the PSM solution, LP resummation is applied to the PSM's truncated series leading to the exact solution. Additionally, the solution procedure does not involve the computation of noise terms. This advantage reduces the computation effort considerably and improves the efficiency of the technique. It is important to note that the high stiffness of these problems was effectively handled by the LPPSM due to the power of PSM and resummation capability of Laplace-Padé. The main advantage of the results of the present work over the pure numerical approaches [5] , [6], [7], [8], [9], [10] is that LPPSM was able to obtain the exact solution without any possibility of round-off errors and its derivative consequences.

Both PSM and LPPSM do not require any initial approximation as the homotopy perturbation method (HPM). Furthermore, the PSM gives the coefficients using an easy straightforward procedure that can be implemented using software packages like Maple or Mathematica. Finally, if the exact solution of an SODEs system is not expressible in terms of known functions then the LP resummation method will enlarge the domain of convergence of the PSM's truncated series.

8 Conclusion

This work presents a technique called LPPSM that combines the PSM and a resummation method based on the Laplace transforms and the Padé approximant. First, the solutions of SODEs systems are obtained in convergent series forms using the

PSM. Then, to enlarge the domain of convergence of the truncated power series solution, a post-treatment using Laplace transform and the Padé approximant is applied. This technique improves PSM's truncated series solutions in convergence rate, and often leads to the exact solution. The PSM is a powerful tool for solving this kind of problems, since it does not require a perturbation parameter or an initial approximation to work and does not generate secular terms (noise terms) as other semi-analytical methods like HPM, HAM or VIM.

By solving three SODEs problems, we presented the LPPSM as a useful tool with high potential to solve linear and nonlinear SODEs systems. Furthermore, we obtained successfully the exact solutions of these three problems showing the efficiency of LPPSM. In addition, the proposed technique is based on a straightforward procedure which is in particular suitable for engineers.

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Competing Interests

Authors have declared that no competing interests exist.

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