British Journal of Mathematics & Computer Science
11(3): 1-6, 2015, Article no.BJMCS.19252
ISSN: 2231-0851

SCIENCEDOMAIN *international*
www.sciencedomain.org

Solutions to the Equations $2^w - 3^n \pm 1=0$ with *w* and *n* Positive Integers

Alain Jaeckel[1*], Jean-François Palierne[2] and Jean Dayantis[3]

[1]*Ecole Européenne de Strasbourg, 70, Boulevard d'Anvers, 67000 Strasbourg, France.*
[2]*Ecole Normale Supérieur de Lyon, Laboratoire de Physique, 69364 Lyon, France.*
[3]*Institut Charles Sadron, Centre de Recherches Sur Les Macromolécules, 67034 Strasbourg, France (Retd.).*

Article Information

DOI: 10.9734/BJMCS/2015/19252
Editor(s):
(1) Dijana Mosic, Department of Mathematics, University of Nis, Serbia.
Reviewers:
(1) Anonymous, Harish-Chandra Research Institute, Allahabad, India.
(2) Anonymous, Université Blaise-Pascal, France.
(3) Anonymous, North University of China, China.
(4) Anonymous, Rajshahi University of Engineering and Technology, Bangladesh.
Complete Peer review History: http://sciencedomain.org/review-history/11190

Short Communication

Received: 31 May 2015
Accepted: 08 August 2015
Published: 31 August 2015

Abstract

We show that the equations $2^w - 3^n \pm 1 = 0$, where *w* and *n* are positive integers, have no other solutions than $(w,n) = (1,0), (1,1), (2,1)$ and $(3,2)$[1].

Keywords: Number theory; Catalan conjecture, harmonical numbers; syracuse-collatz conjecture; unsolved arithmetic problems; Jeffrey C. Lagarias.

1 Introduction

The Belgian mathematician Eugène C. Catalan conjectured in 1844 that $3^2 - 2^3 = 1$ was the only non-trivial solution to the Diophantine equation $x^m - y^n = \pm 1$ (m, $n > 1$) [1,2]. The proof of this conjecture, due to Preda Mihăilescu, was published in 2004 [3].

Long before Catalan's conjecture, in 1343, Levi ben Gershon was interested in studying the pairs of harmonical numbers (in form $2^w 3^n$) differing by ±1 [4]. He solved the equations $2^w - 3^n = \pm 1$ and gave the four solutions in (w,n): (1,0), (1,1), (2,1) and (3,2) [5].

[1]*This result and most of the relationships given in this paper have been suggested by computer calculations used as an investigation tool.*

**Corresponding author: E-mail: alain.jaeckel@ac-strasbourg.fr, jeandayantis@aol.com;*

We have studied the equations $2^w - 3^n \pm 1 = 0$ which appear in a treatment[2] of the so-called Syracuse problem[3,4], more specifically regarding the possibility of cycles with $w > n > 0$ [6-14]. Solving these equations in a more general way, i.e. even for $0 \leq w \leq n$, our aim is to give an original alternative to Levi's proof.

To tackle this problem, we shall focus on the divisibility by 2 of $3^n \pm 1$.

2 Discussion

2.1 Case $w \leq n$

Two solutions to equation

$$2^w - 3^n \pm 1 = 0 \quad (1)$$

appear up to $w = 1$:

$$(w,n) = (1,0): 2^1 - 3^0 - 1 = 2 - 1 - 1 = 0 \quad (2)$$

$$(w,n) = (1,1): 2^1 - 3^1 + 1 = 2 - 3 + 1 = 0 \quad (3)$$

Beyond, for $2 \leq w \leq n$, both $3^n - 1$ and $3^n + 1$ outrun 2^w, precluding any solution in this range.

2.2 Case $3 \geq w > n$

Direct inspection reveals two solutions for $w \leq 3$, namely

$$(w,n) = (2,1): 2^2 - 3^1 - 1 = 4 - 3 - 1 = 0 \quad (4)$$

$$(w,n) = (3,2): 2^3 - 3^2 + 1 = 8 - 9 + 1 = 0 \quad (5)$$

[2] *Unpublished work by the authors of the present article.*

[3] *The so-called Syracuse conjecture has been introduced by Lothar Collatz (1910-1990) in 1937 in Germany and has since then been examined by numerous distinguished mathematicians, including the celebrated Polish mathematician and physicist Stanislaws Ulam (1909-1984), the Japanese Shizuo Kakutani (1911-1004)), the American Jeffrey C. Lagarias ((1949-) and the British born John H. Conway (1937-) now at the Princeton University, USA, as the successor at the chair of John von Neumann.*

The Syracuse conjecture is defined as follows: Let x_0 be a positive and odd integer, and define $x'_1 = 3x_0 + 1$ which is even, and divide it as many times as necessary by 2 to obtain a new odd integer x_1; apply to x_1 the same procedure to obtain a new odd integer x_2 and so on. Collatz conjectured that whatever the starting point x_0, after a varying number of this transformation (or steps), the "flight" always ends in the "trap" 1-4-2-1, with no possibility to escape from.

Since the advent of powerful calculators in the second half of the 20^{th} century, this conjecture has constantly been verified, and therefore it is better today to say the Syracuse-Collatz problem rather than conjecture. However, theoretical proof or at least explanation for this behaviour is not to date available, despite the efforts of so many mathematicians.

[4] *The Syracuse problem in ref. 1 has been generalized to negative odd numbers x_0. Computer calculations show that down to -2.15×10^9, there are here three such "traps", instead of one for positives x_0. These are (we don't show the intermediate even numbers): –1, –2, –1; –5, –7, –5; and –17, –25, –37, –55, –41, –61, –91, –17.*

Whatever the case, though some progress has been made by different workers and also, it is hoped, in ref. 1, the problem remains to date theoretically unsolved.

One may surmise the existence of solutions with $w > 3$, such that $w \approx n$ log3/log2 $\approx 1.58\ n$ for not small w and n. We prove in the following that there is no such solution.

2.3 Case *w* > 3 and *w* > *n*

Defining $\exp_p(n) = \max\{k \in \mathbb{N} \mid p^k$ is a divisor of $n\}$, we will establish two lemmas.

Lemma 1

$$\exp_2(3^n+1) = \begin{cases} 1 \text{ if } n \text{ is even} \\ 2 \text{ if } n \text{ is odd} \end{cases} \tag{6}$$

Lemma 2

$$\exp_2(3^n-1) = \begin{cases} 1 \text{ if } n \text{ is odd} \\ \exp_2(n)+2 \text{ if } n \text{ is even and } n>0 \end{cases} \tag{7}$$

Proof of Lemma 1

An inductive proof is as follows.

Assume that, for a given positive integer i, one has

$$3^{2i} + 1 = 2a, \text{ with odd } a \tag{8}$$

and

$$3^{2i+1} + 1 = 4b, \text{ with odd } b \tag{9}$$

We must prove that this also holds for $i + 1$. One has indeed

$$3^{2(i+1)} + 1 = 2a', \text{ with odd } a' = 9a - 4 \tag{10}$$

and

$$3^{2(i+1)+1} + 1 = 4b', \text{ with odd } b' = 9b - 2 \tag{11}$$

Since relations (8) and (9) hold for $i = 0$ (with $a = b = 1$), they hold for any integer $i \geq 0$, *Q.E.D.*

Proof of Lemma 2

Any positive integer n can be cast in the form

$$n = 2^m k, \text{ with odd } k \tag{12}$$

(k will represent an odd number in all this proof of lemma 2) and therefore we can write

$$3^{2^m k} - 1 = a(k,m)\, 2^{b(k,m)}, \text{ with odd } a \tag{13}$$

Let us consider at first the case $m = 0$. Using (11), we have: $3^k - 1 = 4b' - 2$ with odd b', thus $3^k - 1 = 2c$, with odd $c = 2b' - 1$. Inserting this in (13) yields

$$\exp_2(3^k - 1) = 1 \quad (14)$$

This is the first part of lemma 2.

Let us now suppose m = 1. Using (11) once again, we have: $3^{2k} - 1 = \left(3^k\right)^2 - 1 = (4b - 1)^2 - 1 = 8b'$, with odd b and odd $b' = b(2b - 1)$. Inserting this in (13) yields

$$\exp_2\left(3^{2^m k} - 1\right) = \exp_2\left(2^m b'\right) + 2 = m + 2 \text{, with } m = 1 \quad (15)$$

Now, assume that equation (13) holds for a given odd k and a given $m \geq 1$ with $b(k,m) = m + 2$.

Then $3^{2^{m+1}k} = \left(3^{2^m k}\right)^2 = \left(a\, 2^{m+2} + 1\right)^2 = a'2^{m+3} + 1$, where a is odd, and so is $a' = a + a^2 2^{m+1}$, therefore $\exp_2\left(3^{2^{m+1}k} - 1\right) = \exp_2\left(2^{m+3}a'\right) + 2 = m + 3$. From equation (15), one has

$$\exp_2\left(3^{2^m k} - 1\right) = m + 2 \text{ for } m \geq 1, \textit{Q.E.D.} \quad (16)$$

Equation $2^w - (3^n + 1) = 0$ (17)

Solutions with $w > 2$ are forbidden by Lemma 1 which establishes that $3^n + 1$ cannot be divised by 8.

Equation $2^w - (3^n - 1) = 0$ (18)

Consider first the case when n is odd. Then, from Lemma 2, we have $2^w = 3^n - 1 = 2k$, with odd k, therefore

$$w = k = 1 \quad (19)$$

contradicting our hypothesis $w > 3$.

Consider now the case when n is even, and let $n = 2^m k$ with odd k and $m > 0$. If n and w are to be solutions to equation (18), we have, applying Lemma 2 to equation (18),

$$2^w = 3^{2^m k} - 1 = a(k,m)\, 2^{m+2} \text{, with odd } a \quad (20)$$

which implies

$$a = 1 \text{ and } w = m + 2 \quad (21)$$

consistent with the hypothesis $w > 3$ made at the beginning of this section.

On the other hand, from $w > n = 2^m k$ with $k \geq 1$ and using equation (18), one gets

$$\log_2 w > m + \log_2 k \geq m = w - 2. \quad (22)$$

The resulting condition

$$1 \leq w \leq 3 \quad (23)$$

contradicts the hypothesis $w > 3$.
There is thus no solution to $2^w - (3^n - 1) = 0$ with $w > 3$.

3 Conclusion

Determining the divisibility by 2 of $3^n + 1$ and of $3^n - 1$ enabled us to prove that the algebraic equations $2^w - (3^n \pm 1) = 0$ have only four non-negative integer solutions, namely $(w,n) = (1,0), (1,1), (2,1)$ and $(3,2)$.

Acknowledgements

We would like to thank the referees and, especially, the referee who made us pay attention to the work of Levi ben Gershon which was unknown to us.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Vlad Copil, Ronica Ioan. A brief history of Catalan's conjecture. An. Univ. Spiru Haret. Ser. Mat.-Inform. 2014;10(1)33–36.

[2] Eugène Charles Catalan. Note extraite d'une lettre adressée à l'éditeur. J. Reine Angew. Math. 1844; 27:192.

[3] Preda Mihăilescu. Primary cyclotomic units and a proof of Catalan's conjecture. J. Reine Angew. Math. 2004;572:167–195.

[4] Levi ben Gershon, De Numeris Harmonicis. 1343. See: I. Peterson. Medieval Harmony; 1999.

[5] Shai Simonson. The mathematics of Levi ben Gershon, the Ralbag. Bekhol Derakhekha Daehu 10, Bar-Ilan University Press. 2000;5-21. Winter.

[6] Crandall RE. On the 3*x* +1 problem. Mathematics of Computation. 1978;32:1281-1292.

[7] Jeffrey C. Lagarias. The 3*x*+1 problem and its generalizations. Amer. Math. Monthly. 1985;92:1-23.

[8] Wirsching G. An impoved estimte concerning 3*n*+1 predecessor sets. Acta Arithmetica. 1993;63: 205-210.

[9] Wirsching GJ. The dynamical system generated by the 3*n*+1 function. Springer Verlag; 1998.

[10] Marc Chamberland. A continuous extension of the 3*x*+1 problem to the real line. Dynamics of Continuous, Discrete and Impulsive Systems. 1996;2:495-509.

[11] Marc Chamberland. An update on the 3*x*+1 problem. Catalan translation: "Una actualizacio del problema 3*x*+1", Butlleti de la Societat Catalana de Mathematiques. The original English text may be found on Interrnet at the link "Marc Chamberlan". 2003;18:19-45.

[12] John H. Conway. On unsettleable mathematical problems. Amer. Math. Monthly. 2013;120:182-198.

[13] Nik Lygeros et Olivier Rozier. Dynamique du problème 3*x*+1 sur la droite réelle" arXiv:1402.1979v1 [Math DS] Feb. 2014;1-16. (French).

[14] Cedric Vilani. Théorème vivant. Grasset, Paris. 2012;185.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/11190