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Approximate Solutions for a Couple of Reaction-diffusion Equations with Self-diffusion

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Abstract

In this paper, a competition model of a reaction diffusion system with self-diffusion has been studied using homotopy perturbation method, variational iteration method and Finite element method FEM (COMSOL package). The traveling wave solutions for this system are found and compared numerically.It was shown that the competition will lead at the end of the winning of one species. The effect of self diffusion is shown in the dispersing of traveling wave solution. Also, it was shown that the solution of finite element method and homotopy perturbation method are convergent to each other compared to the variation iteration method.

Keywords: Reaction-diffusion system; Homotopy Perturbation Method; variational iteration method; finite element method.

1 Introduction

Most of problems and phenomena in different fields of science occur nonlinearly. A reaction diffusion equation has applications in fields of ecology, physics, chemistry and biology [1]. Nonlinearity of

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both reaction and diffusion terms describe phenomena in the real life. In ecology, the term ”species ” is used to represent different species of animals, microbes, plants and etc. The species is assumed to move in a random manner (modeled via the diffusion term). The competition between two species is one of the important phenomena in ecology that can be described mathematically as a reaction diffusion system. The competition between species is for resources, which can be for food, as in the competition between red and grey Squirrels for same food, or between Ants and rodents for seeds see [1]. In [18], the competition was between normal and malignant cells, where the common resource is considered to be oxygen. The nonlinear reaction diffusion system (1.1,1.2) has the form,

$$u_t = S_1(u)u_{xx} + F(u,v) \tag{1.1}$$

$$v_t = S_2(v)v_{xx} - G(u,v), \tag{1.2}$$

subject to the initial conditions

$u(x,0) = f(x)$ and $v(x,0) = g(x)$,

and the boundary conditions

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0$,

which are are zero flux boundary conditions. The functions u and v in ecology represent two competing species, $F(u,v) = u(1-u-v)$ and $G(u,v) = -uv$ are the is reaction terms. The rest is the diffusion term, the term $1-u$ in the first equation is a logistic growth, and $-uv$ describes a competition between two species in both equations (the minus sign in $-uv$ means that both species have a negative effect from competition). The self diffusion in (1.1,1.2) is the quadratic functions $S_1(u)$ and $S_2(v)$ multiplying by the diffusion term. The term self diffusion implies the movement of individuals from a higher to lower concentration region (see for example [3], [2]). When $S_1(u) = S_2(v) = 1$, (1.1,1.2) reduced to a simple Noyes-Field model for the Belousov-Zhabotinskii reaction which is studied widely, see for example [4]. Equilibrium solutions are solutions that satisfied $F(u,v) = G(u,v) = 0$, which are here $(u,v) = (0,0)$, $(u,v) = (1,0)$ and $(u,v) = (0,1)$. Traveling wave solution is a solution that connects stable equilibrium solution to other equilibrium solution. It describes how two species (here u and v) are compete for resources. We will focus on the traveling wave that connects two single species, stable equilibrium point $(0,1)$ to $(1,0)$. This kind of wave also called wave fronts (one of the species is the invader) shows that at the end one of the species is winning and the other extincting.

Finite element package COMSOL has been used widely to find the approximate solutions for non linear partial differential equations which has no exact solutions. This tool give a high accuracy of solutions in one and two dimensions (for more details see [5]). This package is used here to solve (1.1,1.2) in one dimension, then we can compare the results from this method to the two methods below.

The homotopy perturbation method [6], proposed first by He in 1998 and was further developed and improved by He [7], [8]. The method yields a very rapid convergence of the solution series in the most cases. Usually, one iteration leads to high accuracy of the solution. Although goal of Hes homotopy Perturbation method was to find a technique to unify linear and nonlinear, ordinary or partial differential equations for solving initial and boundary value problems.

The variational iteration method was first proposed by He (see for example [11],[9],[10]) and was successfully applied to autonomous ordinary differential equations in [12], nonlinear polycrystalline solids in [13], and other fields. The combination of a perturbation method, variational iteration method, method of variation of constants and averaging method to establish an approximate solution of one degree of freedom to weakly nonlinear system in [14]. The variational iteration method has many merits and has much advantages over the Adomian method [15].

The aim of this paper is to obtain and describe the approximate traveling wave solutions for the competition model or the reaction-diffusion system using HPM, VIM and FEM. The capability, effectiveness and convenience of these methods are revealed by obtaining the approximate solutions of the models and comparing it to FEM using COMSOL.

2 Analysis of the Homotopy Perturbation Method

In this section, we illustrate the basic idea of HPM method, first we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \qquad r \in \Omega \tag{2.1}$$

with the equation of boundary

$$B(u, \frac{\partial u}{\partial n}) \qquad r \in \Gamma \tag{2.2}$$

where A is a general differential operator, B a boundary operator, $f(r)$ is known as an analytic function, and Γ is the boundary of the domain Ω. The term $A(u)$ can be divided into two parts which are L and N, where L is a linear operator and is nonlinear term. We can rewrite (2.1) as follows:

$$L(u) + N(u) - f(r) = 0, \qquad r \in \Omega \tag{2.3}$$

Homotopy perturbation structure is shown as follows:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \tag{2.4}$$

where

$$v(r,p) : \ \Omega \times [0,1] \to R \tag{2.5}$$

In (2.4), $p \in [0,1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary condition. Thus, the solution of (2.4) can be written as a power series in p,

$$v = v_0 + pv_1 + p^2 v_2 + \ldots, \tag{2.6}$$

or as an approximate solution,

$$u = \lim_{p \to 0} v = v_0 + v_1 + v_2 + \ldots \tag{2.7}$$

It was shown that the series (2.7) is convergent for most of the cases and the rate of convergence depends on $L(u)$ (see for example [6]). We use this method in the section of result to find the solution of the reaction-diffusion system (1.1) and (1.2).

3 Basic Idea of Variational Iteration Method (VIM)

To clarify the basic ideas of VIM, we consider the following differential equation

$$Lu + Nu = g(x,t), \tag{3.1}$$

where L is a linear operator defined by $L = \frac{\partial^m}{\partial t^m}, m \in \mathbb{N}$, N is a nonlinear operator and $g(x,t)$ is known to be analytic function. The idea of VIM is as follows:

$$\begin{aligned} u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(Lu_n(x,\tau) + N\tilde{u}_n(x,\tau) \\ - g(x,\tau))\, d\tau, \end{aligned} \tag{3.2}$$

where λ is a general Lagrangian multiplier [16] which can be defined as,

$$\lambda(x,\tau) = \frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1}, \;\; m \geq 1. \tag{3.3}$$

The subscript n indicates the nth approximations and $\tilde{u}_n$ is considered as a restricted variation [17]. We used this method to find the approximate solution for (1.1) and (1.2), and compare it to the other methods.

4 Results

In this section, we solve (1.1) and (1.2) using FEM, HPM and VIM, and we find the traveling wave solutions when $S_1(u)$ and $S_2(v)$ are nonlinear terms.

We consider a system of reaction-diffusion equation with self diffusion in the form,

$$u_t = u^2 u_{xx} + u(1-u-v) \tag{4.1}$$
$$v_t = v^2 v_{xx} - uv \tag{4.2}$$

Subject to the initial conditions of:

$$u(x,0) = \frac{e^{kx}}{(1+e^{0.5kx})^2} \tag{4.3}$$
$$v(x,0) = \frac{1}{(1+e^{0.5kx})} \tag{4.4}$$

Where k is constant. The ecological situation we consider from initial conditions, species u is native and species v is introduced locally. The far field boundary conditions are therefore $u \to 1$ and $v \to 0$ as $x \to \infty$. So far we have assumed that in this model one species is native (species u here could be for example Ants) and we introduce new species which could be the invader (species v, for example the Rodents). This is define as a competition for food resources, and can be described by the traveling wave solution that connect the equilibrium solutions as we mentioned before. Since the single equilibrium solution $(0, 1)$ is stable, we expect that species v will win.

4.1 HPM Solutions of (4.1) and (4.2)

In order to solve equation(4.1) and equation(4.2), using HPM, we construct the following homotopy process

$$H(u,p) = (1-p)u_t + p(u_t - u(1-u-v) - u^2 u_{xx}) = 0 \tag{4.5}$$
$$H(u,p) = (1-p)v_t + p(v_t - v^2 v_{xx} + uv) = 0 \tag{4.6}$$

Substituting v from (2.6) into (4.5) and (4.6) and rearrange it basing on powers of p, we can obtain:

$$p^0 : \begin{cases} \frac{\partial u_0}{\partial t} = 0 \\ \frac{\partial v_0}{\partial t} = 0 \end{cases} \tag{4.7}$$

$$p^1 : \begin{cases} \frac{\partial u_1}{\partial t} - u_0 + u_0 v_0 + u_0^2 - u_0^2 \frac{\partial^2 u_0}{\partial x^2} = 0 \\ \frac{\partial v_1}{\partial t} + u_0 v_0 - v_0^2 \frac{\partial^2 v_0}{\partial x^2} = 0 \end{cases} \tag{4.8}$$

$$p^2 : \begin{cases} \frac{\partial u_2}{\partial t} - u_1 + 2u_0u_1 + v_0u_1 + u_0v_1 \\ \qquad -(u_0^2 \frac{\partial^2 u_1}{\partial x^2} + 2u_0u_1 \frac{\partial^2 u_0}{\partial x^2}) = 0 \\ \frac{\partial v_2}{\partial t} + u_0v_1 + v_0u_1 - (v_0^2 \frac{\partial^2 v_1}{\partial x^2} \\ \qquad +2v_0v_1 \frac{\partial^2 v_0}{\partial x^2}) = 0 \end{cases} \tag{4.9}$$

Solving equations (4.7)-(4.9), we obtain:

$$u_0 = \frac{e^{kx}}{(1 + e^{0.5kx})^2} \tag{4.10}$$

$$v_0 = \frac{1}{(1 + e^{0.5kx})} \tag{4.11}$$

$$u_1 = \frac{te^{\frac{3kx}{2}} A}{2\left(e^{\frac{kx}{2}} + 1\right)^8} \tag{4.12}$$

$$v_1 = \frac{\frac{1}{4}\left(k^2 - 4\right) te^{kx} - \frac{1}{4} te^{\frac{kx}{2}} B}{\left(e^{\frac{kx}{2}} + 1\right)^5} \tag{4.13}$$

$$u_2 = \frac{t^2 e^{\frac{3kx}{2}} C}{16\left(e^{\frac{kx}{2}} + 1\right)^{14}} \tag{4.14}$$

$$v_2 = \frac{t^2 e^{x/2} D}{32\left(e^{x/2} + 1\right)^9} \tag{4.15}$$

$$\vdots$$

where
$A = 2k^2e^{\frac{3kx}{2}} - k^2e^{2kx} + 8e^{\frac{kx}{2}} + 8e^{\frac{3kx}{2}} + 12e^{kx} + 2e^{2kx} + 2.$

$B = \left(k^2 + 4e^{\frac{3kx}{2}} + 8e^{kx}\right).$

$C = 448e^{6kx} + 2k^2 + 28k^2e^{\frac{3kx}{2}} - k^2\left(k^2 - 6\right)e^{5kx}$
$+ 4\left(3k^2 + 4\right)e^{\frac{kx}{2}} + 14\left(3k^2 + 64\right)e^{2kx} + 20\left(5k^2 + 56\right)e^{\frac{5kx}{2}}$
$+ 14\left(5k^2 + 64\right)e^{3kx} + 4\left(7k^2 + 32\right)e^{kx}$
$+ \left(-129k^4 - 84k^2 + 128\right)e^{4kx} + 2\left(17k^4 - 8k^2 + 8\right)e^{\frac{9kx}{2}}$
$+ \left(88k^4 - 60k^2 + 448\right)e^{\frac{7kx}{2}}.$

$D = 3e^{x/2} - 45e^x + 11e^{\frac{3x}{2}} - 24e^{2x} + 56e^{3x} + 16e^{\frac{7x}{2}} - 1.$

The solution of (4.1) and (4.2), when $p \to 1$, will be as follows:

$$u(x, t) = u_0 + u_1 + u_2 + \dots. \tag{4.16}$$

$$v(x, t) = v_0 + v_1 + v_2 + \dots. \tag{4.17}$$

4.2 VIM Solutions of (4.1) and (4.2)

In order to solve (4.1) and (4.2), using VIM, we construct a correction functional

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda((u_n)_\tau - u_n(1-u_n-v_n) - u_n^2(u_n)_{xx})\, d\tau. \quad (4.18)$$

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda((v_n)_\tau + u_n v_n - v_n^2(v_n)_{xx})\, d\tau. \quad (4.19)$$

In our equation $m = 1$, then from the formula (3.3) and $\lambda = -1$, substituting it in (4.18) and (4.19) we get:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t ((u_n)_\tau - u_n(1-u_n-v_n) - u_n^2(u_n)_{xx})\, d\tau. \quad (4.20)$$

$$v_{n+1}(x,t) = v_n(x,t) - \int_0^t ((v_n)_\tau + u_n v_n - v_n^2(v_n)_{xx})\, d\tau. \quad (4.21)$$

For $n = 0, 1, \ldots$.

We start with the initial approximations of $u(x,0)$ and $v(x,0)$ given by (4.3) and (4.4). Using the iterations (4.20) and (4.21), we can obtain the other components as follows:

$$u_0(x,t) = u(x,0) = \frac{e^{kx}}{(1+e^{0.5kx})^2}. \quad (4.22)$$

$$v_0(x,t) = v(x,0) = \frac{1}{(1+e^{0.5kx})} \quad (4.23)$$

For $n = 0$

$$u_1(x,t) = u_0(x,t) - \int_0^t ((u_0)_\tau - u_0(1-u_0-v_0) - u_0^2(u_0)_{xx})\, d\tau,$$

$$u_1(x,t) = \frac{te^{\frac{3kx}{2}}E}{2\left(e^{\frac{kx}{2}}+1\right)^8} + \frac{e^{kx}}{\left(e^{\frac{kx}{2}}+1\right)^2}. \quad (4.24)$$

$$v_1(x,t) = v_0(x,t) - \int_0^t ((v_0)_\tau + u_0 v_0 - v_0^2(v_0)_{xx})\, d\tau,$$

$$v_1(x,t) = \frac{\frac{1}{4}\left(k^2-4\right)te^{kx} - \frac{1}{4}te^{\frac{kx}{2}}F}{\left(e^{\frac{kx}{2}}+1\right)^5} + \frac{1}{e^{\frac{kx}{2}}+1}. \quad (4.25)$$

where

$E = 2k^2e^{\frac{3kx}{2}} - k^2e^{2kx} + 8e^{\frac{kx}{2}} + 8e^{\frac{3kx}{2}} + 12e^{kx} + 2e^{2kx} + 2.$

$F = \left(k^2 + 4e^{\frac{3kx}{2}} + 8e^{kx}\right).$

By the same way we can get $u_2(x,t), u_3(x,t), \ldots$.

The solution of (4.1) and (4.2) via VIM will be in the form follows:

$$u(x,t) = u_n(x,t). \tag{4.26}$$

$$v(x,t) = v_n(x,t). \tag{4.27}$$

The traveling wave solutions of $u(x,t)$ and $v(x,t)$ which are obtained by HPM, VIM, and FEM are shown in (Figs. 1 to 4). It can be deduced that the traveling wave solutions which obtained from the three methods are more consistent when $t = 0.5$. When time is increased, the traveling wave solutions which obtained from HPM and FEM are more consistent, while the solutions we get from VIM are not entirely agreed with the above methods.

The ecological meaning of the results which are obtained is; we have an invasion of one species namely v on species u, and this invasion effects on the resources that were exists before the invasion. When time increased the invader is winning. This is what we expect from the nature of equilibrium solutions and the conditions in the model we have assumed.

4.3 Model with and Without Self-diffusion

In this section, we compare the traveling wave solution of (1.1,1.2) with and without self diffusion, in other wards, when $S_1(u) = S_2(v) = 1$ the model in (1.1,1.2) turns out to be without self-diffusion. FEM is used to compare the solution in both cases as shown in (Figs. 5 and 6). It can be seen how self-diffusion effects on disperse of traveling wave solution of u and v, and has no effect on which species will win at the end.

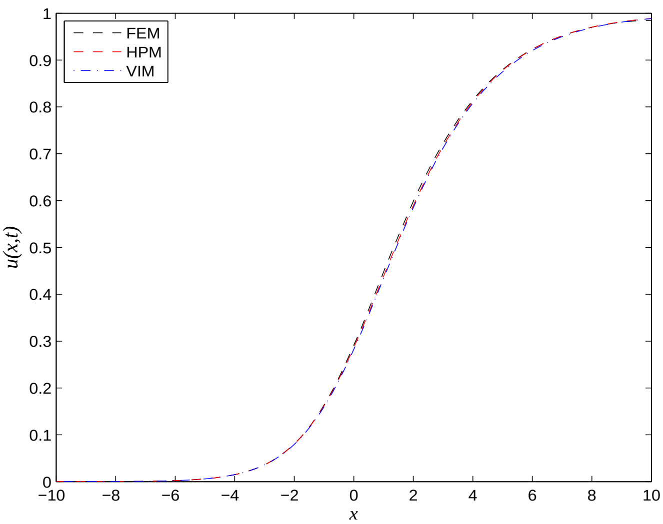

Fig. 1. Comparison between the traveling wave solution of $u(x,t)$ using VIM, HPM and FEM, when $t = 0.5$

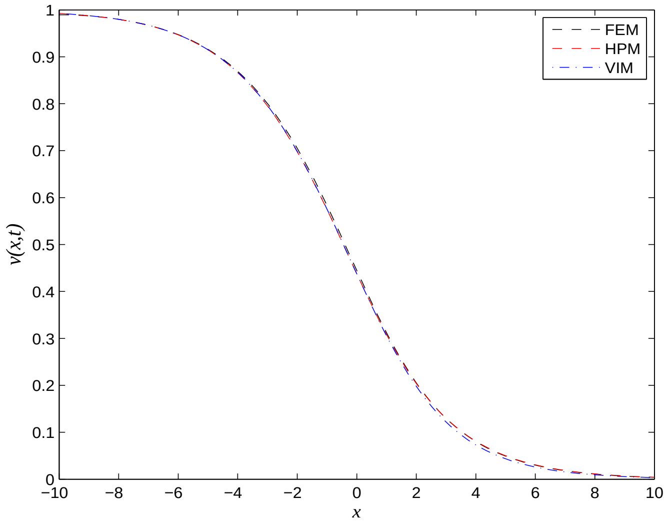

Fig. 2. Comparison between the traveling wave solution of $v(x,t)$ using VIM, HPM and FEM, when $t = 0.5$

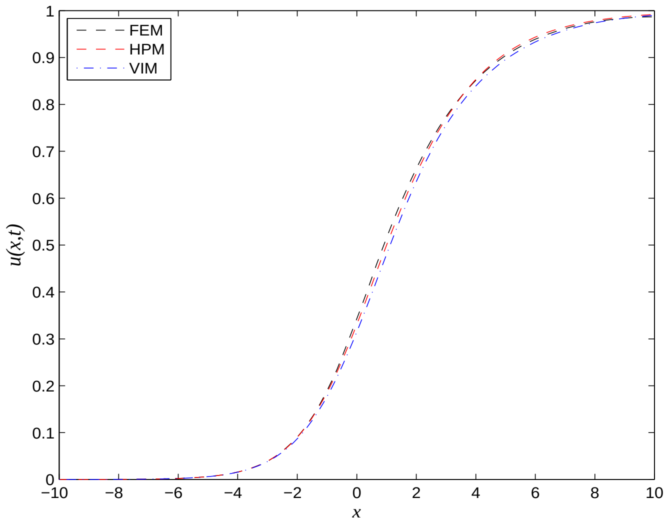

Fig. 3. Comparison between the traveling wave solution of $u(x,t)$, using VIM, HPM and FEM, when $t = 1$

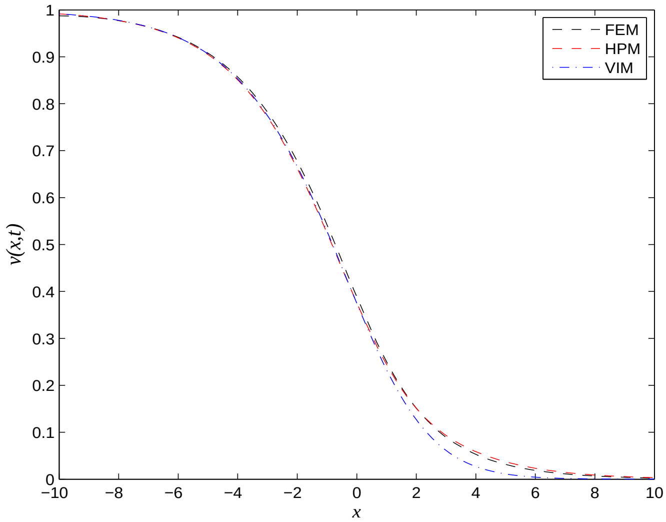

Fig. 4. Comparison between the traveling wave solution of $v(x,t)$, using VIM, HPM and FEM, when $t = 1$

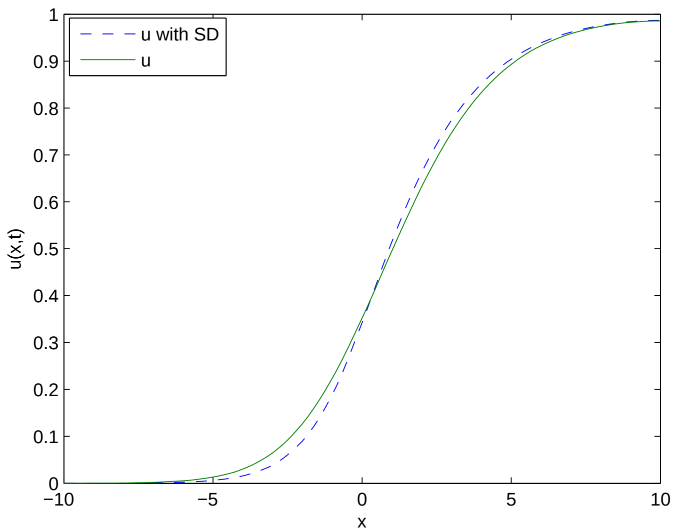

Fig. 5. Comparison between traveling wave solution of $u(x,t)$, with and without self-diffusion when $t = 1$ and using COMSOL

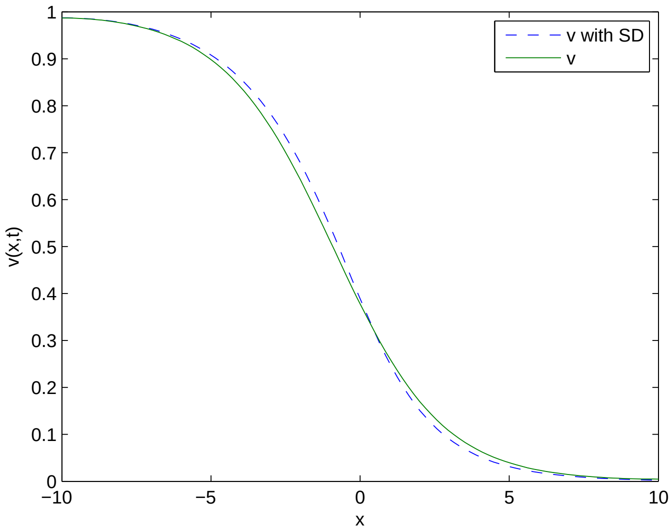

Fig. 6. Comparison between traveling wave solution of $v(x,t)$, with and without self-diffusion when $t = 1$ and using COMSOL

5 Conclusion

In conclusion, three powerful numerical methods are used to find the traveling wave solutions of reaction-diffusion model with self-diffusion in one dimension. It was shown that the traveling wave solutions which are gets from FEM and HPM are more consistent compare to the solutions get from VIM with variety in times.

Ecologically, we have shown that from results how two species compete for resources, and how one of the species will win at the end when one of the species is the invader (species v). The result of traveling solutions are agreed with the prediction we have made. The effect of the self-diffusion on the solutions is shown and it can be deduced how the solutions will disperse and spread out when the degree of nonlinearity in self-diffusion is changed.

Competing Interests

The authors declare that no competing interests exist.

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