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A Model of Elasticity Taking into Account the Displacement Orientation in the Deformation

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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Short Research Article

Abstract

Aims/Objectives: In this work, we deal an elasticity model in 2D and 3D dimension for deformation under constraint by taking into account the deformation displacement orientation. This mathematical model can be used, for example, to describe the heart deformation taking into account the orientation of the fibers for estimating global and regional parameters of the left ventricular function.
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Methodology: In first, we start by studying a model of Poisson problem under constraint on a domain $\Omega \subset \mathbb{R}^n$ (n=2 or 3), considering a constraint on a part K of this domain. Secondly, we consider the proposed 2D and 3D elasticity model of deformation under constraint by taking into account the displacement orientation in the deformation. We treat only the case where the orientation of displacement is a given constant vector. The eigenvalues case is being finalized.

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A major difficulty for those problems is to find a demonstration of existence and uniqueness of solution, which are given in both 2D and 3D dimension.
A numerical approach of the solution using finite element method and its convergence is studied. Numerical simulations are performed with Free Fem software. Simulations results and comments are given in the end.
Results: Existence and uniqueness results are established in Sobolev spaces for the proposed models. Convergence of the FEM approached solution is given and numerical simulations are performed with success.
Conclusion: This work has been devoted to study a problem of elasticity under constraint in order to take account the orientation of structure displacement. Both analytical and numerical study of the proposed problem are realized. The numerical simulations give good results.

Keywords: Elasticity; orientation; deformation; simulation; existence; uniqueness; convergence.

1 Introduction

The heart is a vital organ whose eventual malfunctions can have fatal consequences, especially cardiovascular disease and ischemic diseases are among the most deadly in industrialized countries. Detection and prevention of such diseases is therefore a major challenge for modern medicine [1].

Actually, the patient's prognosis with myocardial infarction is dependent not only on the location and size of the infarction but also on the precise evaluation of the residual myocardial viability which ideally allow to define a therapeutic strategy well suited.

Imaging modalities (scintigraphy, echocardiography, X-ray computed tomography, magnetic resonance imaging (MRI) and ultrasonic imaging) were specified for acquiring images of the heart in motion in two or three spatial dimensions.

The diagnostic medical imaging is mainly based on a visual analysis of segmental contractile function and manual segmentation or semiautomatic for estimating global and regional parameters of the left ventricular function [2].

This segmentation remains a time-consuming task, and of course subject to considerable variability within and between operators. Research in the field of medical image processing aimed at making the process of analyzing specific images faster and more automatic to reduce subjectivity and time required for implementation of the algorithms [3],[4].

A semi-automatic, or automatic, segmentation process requires a good mathematical model for expressing the deformation of the tissues. In that context, it is known, that a wrong stress deformation model leads to a bad heart segmentation, in term of contours definition [5]. The heart tissues are made of fibers which impose privileged directions in which the deformations take place [6].

At the moment, mathematical models failed to represent the fibers of the heart tissues, therefore, we propose to introduce a constraint on the feasible direction for the stress deformation in order to mimic the fiber orientation.

The aim of this work is twofold: first to propose, a mathematical elastic model in 2D and 3D subject to a stress deformation constraint which mimics the fiber orientation in the left ventricular of the heart. And second to prove this model has one solution which can be approximated with a finite element method.

In order to keep the presentation as clear as possible, the first section is dedicated to a simplified situation where a diffusion model is substituted to an elastic model. Thus, the displacement u solves a Poisson problem subject to a constraint for its gradient. Then a mixed formulation is given for the Poisson problem subject to a constraint and an existence and uniqueness result is discussed for that formulation.

In the second section, we consider that the displacement u satisfies a linear elastic model subject to a constraint which is enforced by the stress fiber orientation. For this mathematical model, the heart is considered as an orthotropic environment [6],[5][7],[8].

As previously, the problem is expressed in a mixed context, leading to a problem more intricated than the one given in the first section. An existence and uniqueness result in 2D and 3D dimension are given. The main difficulty for those problems is to find an equivalent problem which leads to a mixed formulation and permits thanks to a saddle point problem to prove existence and uniqueness result.

The proposed models are studied numerically and an a priori error estimate is established. The paper is ended with numerical simulations, which are performed with Free Fem software [9]. The effects of the constraint is assessed with these numerical simulations.

2 A Poisson Problem Under Constraint of Strain

Neglecting the elastic nature of the heart tissues, the displacement of a slice of the left ventricle of the heart, denoted by Ω, is denoted by u. Let w be a privileged direction for the displacement gradient, then u solves a Poisson problem subject to the constraint $\nabla u.w = 0$ [10].

In the sequel, Ω is an open bounded domain of $\mathbb{R}^n$ (n=2 or 3), whose boundary is denoted by Γ.

2.1 The mathematical model

We denote by u the vector describing the displacement of a material point x in Ω, f expresses the exerted pressure on Ω, wherein the constraint is verified.

The simplified model taking into account the constraint of gradient leads to the following equations:

$$\begin{cases} -\Delta u = f & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \\ \nabla u.w = 0 & on\ \Omega. \end{cases} \tag{2.1}$$

where w is a given displacement vector direction such that $w = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, and $(a_1, ..., a_n) \in \mathbb{R}^n$.

In what follows, basically, a mixed formulation for the Poisson problem consists in dealing with two unknowns, the displacement and the gradient of the displacement.

2.2 Weak mixed formulation

Let $V = (L^2(\Omega))^{n\times n}$ and $W = \left\{ v \in (H_0^1(\Omega))^n / \nabla u.w = 0 \text{ on } \Omega \right\}$. Then W equipped with the inner product $\langle .,. \rangle_{(H_0^1(\Omega))^n}$ is a non empty closed vector subspace in $(H_0^1(\Omega))^n$. We set $\sigma = -\nabla u$, and we consider the following problem:

$$\begin{cases} Find\ (\sigma,u)\in V\times W \text{ verifying:} \\ \sigma+\nabla u=0 \qquad \text{in } \Omega, \\ div\sigma=f \qquad \text{in } \Omega, \\ u=0 \qquad \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

u the displacement vector, f the exerted pressure on Ω, σ is the variation of u.

Remark 2.1. The problem (2.1) is equivalent to (2.2).

Let $\tau \in V$ and $v \in W$ two test functions, then

$$\begin{cases} \int_\Omega \sigma\times\tau dx+\int_\Omega \nabla u\times\tau dx=0 \qquad \forall\tau\in V, \\ \int_\Omega div\sigma.vdx=\int_\Omega f.vdx \qquad \forall v\in W. \end{cases} \tag{2.3}$$

with $\times$ is the matrix product in $M_{n\times n}(\mathbb{R})$.

Using Green formula in the second equation and adding to the first one, we obtain

$$\int_\Omega \sigma\times\tau dx+\int_\Omega \nabla u\times\tau dx+\int_\Omega \sigma\times\nabla vdx=-\int_\Omega f.vdx \qquad \forall\tau\in V\text{and} \quad v\in W. \tag{2.4}$$

For $(\sigma,\tau,v)\in V\times V\times W$, we put

$$a(\sigma,\tau)=\int_\Omega \sigma\times\tau dx=\int_\Omega \sum_{i,j=1}^{n}\sigma_{ij}\tau_{ji}dx$$

and

$$b(\sigma,v)=\int_\Omega \sigma\times\nabla vdx=\int_\Omega \sum_{i,j=1}^{n}\sigma_{ij}\frac{\partial v_j}{\partial x_i}dx$$

Proposition 2.1. *a(.,.) is a bilinear, symmetric and continuous form on $V\times V$, and b(.,.) is a bilinear and continuous form on $V\times W$. Moreover a(.,.) is coercive.*

Proof. We know that a(.,.) is a bilinear and symetric form. On the other hand we have,

$$a(\sigma,\tau)=\int_\Omega \sigma\times\tau dx=\int_\Omega \sum_{i,j=1}^{n}\sigma_{ij}\tau_{ij}dx$$

thanks to Hlder and Minkowski inequalities [11], we obtain:

$$|a(\sigma,\tau)|\leq\left(\sum_{i,j=1}^{n}\int_\Omega |\sigma_{ij}|^2dx\right)^{\frac{1}{2}}\times\left(\sum_{i,j=1}^{n}\int_\Omega |\tau_{ij}|^2dx\right)^{\frac{1}{2}}$$

then

$$|a(\sigma,\tau)|\leq||\sigma||_V||\tau||_V$$

so a(.,.)is a bilinear, symmetric and continuous form. Since

$$a(\sigma,\sigma)=\int_\Omega \sigma\times\sigma dx=||\sigma||_V^2.$$

We demonstrate that a(.,.) is a bilinear, symmetric, continuous and coercive form on $V \times V$.
Using the same steps for the demonstration of continuity of the bilinear form a(.,.), we have:

$$|b(\sigma, v)| \leq ||\sigma||_V ||\nabla v||_V \leq ||\sigma||_V ||v||_W$$

so b(.,.) is a bilinear and continuous form on $V \times W$. □

In order to put the problem (2.2) in the context of a mixed formulation, we consider the following system

$$\left\{ \begin{array}{l} \text{find } \sigma \in V \text{ verifying :} \\ a(\sigma, \tau) = 0 \qquad \forall \tau \in V. \end{array} \right. \tag{2.5}$$

and satisfying the additional constraints

$$b(\tau, v) = \int_\Omega f.vdx \quad \text{for all} \quad v \in W. \tag{2.6}$$

since V and W are Hilbert spaces and thanks to the proposition (2.1) ,this problem is equivalent to find $\sigma \in V$ such that:

$$J(\sigma) = min \left\{ J(\tau)/\tau \in V \text{ and } b(\tau, v) = -\int_\Omega f.vdx, \forall v \in W \right\}.$$

where

$$J(\tau) = \frac{1}{2} a(\tau, \tau); \forall \tau \in V.$$

Then the problem (2.5) under constraints (2.6) has been transformed into a minimization problem under constraints.

We introduce now the following Lagrangian:

$$L(\tau, v) = J(\tau) + b(\tau, v) - G(v)$$

with

$$G(v) = -\int_\Omega f.vdx.$$

Notice that

$$\left\langle \frac{\partial L}{\partial \tau}(\sigma, u), \tau \right\rangle_V = a(\sigma, \tau) + b(\tau, u) = 0$$

and

$$\left\langle \frac{\partial L}{\partial v}(\sigma, u), v \right\rangle_W = b(\sigma, v) - G(v) = 0.$$

$\langle ., . \rangle_V$ and $\langle ., . \rangle_W$ are respectively the inner product in V and W. So a weak mixed formulation of the problem (2.2) is given by the following system [10]:

$$\left\{ \begin{array}{ll} \text{find}(\sigma, u) \in V \times W \text{verifying :} & \\ a(\sigma, \tau) + b(\tau, u) = 0 & \forall \tau \in V, \\ b(\tau, v) = G(v) & \forall v \in W. \end{array} \right. \tag{2.7}$$

2.3 Existence and uniqueness theorem

We give in this paragraph, a result of existence and uniqueness based on Brezzi's theorem [10],[12] for the solution of the Poisson model under a constraint of deformation.

Theorem 2.1. *If the bilinear form a(.,.) is symmetric, continuous and coercive on $V \times V$, and the bilinear form b(.,.) is continuous on $V \times W$, then if there exist a real $\gamma > 0$ satisfying the following condition "$inf sup$":*

$$\sup_{\tau \in V}\left(\inf_{v \in W} \frac{b(\tau, v)}{\|v\|_W \|\tau\|_V}\right) \geq \gamma$$

the problem (2.7) admits unique solution.

Proof. Thanks to proposition (2.1), we know that the bilinear form a(.,.) is symmetric and continuous on $V \times V$, and the bilinear form b(.,.) is continuous on $V \times W$. Moreover a(.,.) is coercive. On the other hand, let choosing $\tau = \nabla v$ for a given $v \in W$, then

$$\frac{b(\tau, v)}{\|v\|_W \|\tau\|_V} = \frac{\int_\Omega \tau \times \nabla v dx}{\|v\|_W \|\tau\|_V} = \frac{\int_\Omega \nabla v \times \nabla v dx}{\|v\|_W \|\tau\|_V} = \frac{\|\nabla v\|_V^2}{\|v\|_W \|\nabla v\|_V} = 1.$$

So

$$\sup_{\tau \in V}\left(\inf_{v \in W} \frac{b(\tau, v)}{\|v\|_W \|\tau\|_V}\right) \geq 1$$

and we deduce that the problem (2.7) is well posed, consequently the Poisson problem under constraint of strain (2.1) admits a unique solution. □

3 An Elasticity Model with Stress Fiber Orientation

In this section, we focus on the case where the displacement u is described by a model of elasticity in both cases 2D and 3D dimension, taking into account the orientation of the fibers in the heart deformation. Let $n \in \{2, 3\}$ and Ω an open bounded domain in $\mathbb{R}^n$.

3.1 The mathematical model

We recall that the strain tensor and the stress tensor are defined as follows:

$$\varepsilon(u) = \begin{pmatrix} \varepsilon_{11}(u) & \varepsilon_{12}(u) & ... & \varepsilon_{1n}(u) \\ \varepsilon_{21}(u) & \varepsilon_{22}(u) & ... & \varepsilon_{2n}(u) \\ : & : & ... & : \\ : & : & ... & : \\ \varepsilon_{n1}(u) & \varepsilon_{n2}(u) & ... & \varepsilon_{nn}(u) \end{pmatrix}$$

where $\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

$$\sigma(u) = \begin{pmatrix} \sigma_{11}(u) & \sigma_{12}(u) & ... & \sigma_{1n}(u) \\ \sigma_{21}(u) & \sigma_{22}(u) & ... & \sigma_{2n}(u) \\ : & : & ... & : \\ : & : & ... & : \\ \sigma_{n1}(u) & \sigma_{n2}(u) & ... & \sigma_{nn}(u) \end{pmatrix}.$$

Given $\theta \in \mathbb{R}$ and $w \in \Omega$, the proposed mathematical model is defined as follows:

$$\begin{cases} -div\sigma(u) = f \quad \text{in } \Omega, \\ \varepsilon(u)w = \theta w \quad \text{in } K \subset \Omega, \\ u = 0 \quad \text{in } \Gamma. \end{cases} \tag{3.1}$$

K is the part of Ω, wherein the constraint is verified and w is a given vector orienting the displacement such that $w = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ and $(a_1, ..., a_n) \in \mathbb{R}^n$.
Before studying this model, we introduce some properties of matrix.

Definition 3.1. Let T a matrix $n^2 \times n^2$ and A a matrix $n \times n$, such that

$$T = \begin{pmatrix}
t_{1111}\ t_{1112}\ ...\ t_{111n}\ t_{1121}\ t_{1122}\ ...\ t_{112n}\ ....\ t_{11n1}\ t_{11n2}\ ...\ t_{11nn} \\
t_{1211}\ t_{1212}\ ...\ t_{121n}\ t_{1221}\ t_{1222}\ ...\ t_{122n}\ ....\ t_{12n1}\ t_{12n2}\ ...\ t_{12nn} \\
. \\ . \\ . \\
t_{1n11}\ t_{1n12}\ ...\ t_{1n1n}\ t_{1n21}\ t_{1n22}\ ...\ t_{1n2n}\ ....\ t_{1nn1}\ t_{1nn2}\ ...\ t_{1nnn} \\
. \\ . \\ . \\
t_{i111}\ t_{i112}\ ...\ t_{i11n}\ t_{i121}\ t_{i122}\ ...\ t_{i12n}\ ....\ t_{i1n1}\ t_{i1n2}\ ...\ t_{i1nn} \\
t_{i211}\ t_{i212}\ ...\ t_{i21n}\ t_{i221}\ t_{i222}\ ...\ t_{i22n}\ ....\ t_{i2n1}\ t_{i2n2}\ ...\ t_{i2nn} \\
. \\ . \\ . \\
t_{in11}\ t_{in12}\ ...\ t_{in1n}\ t_{in21}\ t_{in22}\ ...\ t_{in2n}\ ....\ t_{inn1}\ t_{inn2}\ ...\ t_{innn} \\
. \\ . \\ . \\
t_{n111}\ t_{n112}\ ...\ t_{n11n}\ t_{n121}\ t_{n122}\ ...\ t_{n12n}\ ....\ t_{n1n1}\ t_{n1n2}\ ...\ t_{n1nn} \\
t_{n211}\ t_{n212}\ ...\ t_{n21n}\ t_{n221}\ t_{n222}\ ...\ t_{n22n}\ ....\ t_{n2n1}\ t_{n2n2}\ ...\ t_{n2nn} \\
. \\ . \\ . \\
t_{nn11}\ {}_{nn12}\ ...\ t_{nn1n}\ t_{nn21}\ t_{nn22}\ ...\ t_{nn2n}\ ....\ t_{nnn1}\ t_{nnn2}\ t...\ t_{nnnn}
\end{pmatrix}. \tag{3.2}$$

One can also write $T = (t_{ijkl})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n \\ 1 \leq k \leq n \\ 1 \leq l \leq n}}$.

Then, we define the following matrix product $T : A = \sum_{ijkl=1}^{n} t_{ijkl} a_{kl}$.

Remark 3.1.

1. $Id_{\mathbb{R}^{n^2 \times n^2}} : B = B$ for all B in $M_{n \times n}(\mathbb{R})$.
2. $(A + T) : B = A : B + T : B$ for all A and T in $M_{n^2 \times n^2}(\mathbb{R})$, and B in $M_{n \times n}(\mathbb{R})$.
3. $A : (T : B) = (A \times T) : B$ for all A and T in $M_{n^2 \times n^2}(\mathbb{R})$, and B in $M_{n \times n}(\mathbb{R})$.

To more simplify the model 3.1, we consider the following system:

$$\begin{cases} -div\sigma(u) = f \quad \text{in } \Omega, \\ \varepsilon(u) = \theta Id_{\mathbb{R}^n} \quad \text{in } K \subset \Omega, \\ u = 0 \quad \text{in } \Gamma. \end{cases} \tag{3.3}$$

Remark 3.2. If u is a solution of (3.3), then u is solution of (3.1).

Since the heart is an orthotropic medium [13],[14],[15],[16] the stress tensor can be written as follows:

$$\sigma(u) = C : \varepsilon(u).$$

C is the elastic stiffness. C is a n^2 rank tensor which can be writen as (3.2) [11].

Knowing that $\sigma(u) = C : \varepsilon(u)$, and that D is the C inverse matrix, witch named the elastic complaisance [17],[18] then the problem (3.3) is equivalent to the following system:

$$\begin{cases} -div\sigma(u) = f \quad \text{in } \Omega, \\ \sigma(u) = C : \theta Id_{\mathbb{R}^n} \quad \text{in } K \subset \Omega, \\ u = 0 \quad \text{in } \Gamma. \end{cases} \tag{3.4}$$

Thanks to remark (3.1), the problem (3.4) is equivalent to the following problem:

$$\begin{cases} -div\sigma(u) = f \quad \text{in } \Omega, \\ \sigma(u) + \varepsilon(u) = (C + Id_{\mathbb{R}^{n\times n}}) : \theta Id_{\mathbb{R}^n} \quad \text{in } K \subset \Omega, \\ u = 0 \quad \text{in } \Gamma. \end{cases} \tag{3.5}$$

We introduce the spaces

$$V = L^2(\Omega)^{n\times n}$$

and

$$W = \left\{ v \in (H_0^1(\Omega))^n / \exists \theta \in \mathbb{R}^n \ \text{ such as } \ \sigma(v) + \varepsilon(v) = (C + Id_{\mathbb{R}^{n\times n}}) : \theta Id_{\mathbb{R}^n} \ \text{ on K} \right\}$$

W equipped with the inner product $\langle ., . \rangle_{(H_0^1(\Omega))^n}$ is a non empty closed vector subspace in $(H_0^1(\Omega))^n$, consequently V and W are Hilbert spaces. We consider the following problem

$$\begin{cases} \text{find } \ (\sigma, u) \in V \times W \text{ verifying :} \\ \sigma(u) + \varepsilon(u) = (C + Id_{\mathbb{R}^{n\times n}}) : \theta Id_{\mathbb{R}^n} \quad \text{in } \ \Omega, \\ -div(\sigma) = f \qquad \text{in } \ \Omega, \\ u = 0 \ \text{ on } \ \partial\Omega. \end{cases} \tag{3.6}$$

Notice that a solution of the problem (3.6) is also a solution of the problem (3.1). Now let $\tau \in V$ and $v \in W$ two test functions, then:

$$\begin{cases} \displaystyle\int_\Omega \sigma \times \tau dx + \int_\Omega \varepsilon(u) \times \tau dx = \int_\Omega (C + Id_{\mathbb{R}^{n\times n}}) : \theta Id_{\mathbb{R}^n} \times \tau dx \qquad \forall \tau \in V \\ \displaystyle -\int_\Omega div(\sigma).v dx = \int_\Omega f.v dx \qquad \forall v \in W \end{cases} \tag{3.7}$$

Using Green formula and adding those two equations, we obtain:

$$\int_\Omega \sigma \times \tau dx + \int_\Omega \varepsilon(u) \times \tau dx + \int_\Omega \sigma \times \varepsilon(v) dx = \int_K f.v dx + \theta \int_\Omega (C + Id_{\mathbb{R}^{n\times n}}) : \theta Id_{\mathbb{R}^n} \times \tau dx \tag{3.8}$$

We consider

$$a(\sigma, \tau) = \int_K \sigma \times \tau dx, \quad \forall (\sigma, \tau) \in V \times V,$$

$$b(\tau, v) = \int_K \tau \times \varepsilon(v) dx, \quad \forall (\tau, v) \in V \times W,$$

$$l(\tau) = \int_\Omega (C + Id_{\mathbb{R}^{n\times n}}) : \theta Id_{\mathbb{R}^n} \times \tau dx, \quad \forall \tau \in V,$$

$$G(v) = \int_K f.v dx, \quad \forall v \in W.$$

Proposition 3.1. *a(.,.) is a bilinear, symmetric, continuous and coercive form on $V \times V$, and b(.,.) is a bilinear and continuous form on $V \times W$.*

Proof. We know that a(.,.) is a bilinear and symmetric form. On the other hand we have,

$$a(\sigma, \tau) = \int_\Omega \sigma \times \tau dx$$

Thanks to Hlder and Minkowski inequalities [11], we obtain:

$$|a(\sigma, \tau)| \leq M||\sigma||_V||\tau||_V.$$

So a(.,.) is a bilinear, symmetric and continuous form. Since

$$a(\sigma, \sigma) = \int_\Omega \sigma \times \sigma dx$$

then:

$$a(\sigma, \sigma) = ||\sigma||_V^2.$$

so a(.,.) is a bilinear, symmetric, continuous and coercive form on $V \times V$. For the bilinearity of b(.,.) is easily verified, let demonstrate that it's continuous.

$$|b(\tau, v)| = \left|\int_\Omega \tau \times \varepsilon(v)dx\right| = \left|\int_\Omega \sum_{i,j=1}^n \tau_{ij}\varepsilon_{ji}(v)dx\right|$$

thanks to Hlder and Minkowski inequalities, we obtain:

$$|b(\tau, v)| \leq \left(\sum_{i,j=1}^n \int_\Omega |\tau_{ij}|^2 dx\right)^{\frac{1}{2}} \left(\sum_{i,j=1}^n \int_\Omega |\varepsilon_{ji}(v)|^2 dx\right)^{\frac{1}{2}}$$
$$\leq ||\tau||_V||\varepsilon(v)||_V \leq ||\tau||_V||v||_W$$

So b(.,.) is a bilinear and continuous form on $V \times W$. □

We consider the following problem:

$$\begin{cases} \text{find } \sigma \in V \text{ verifying :} \\ a(\sigma, \tau) = l(\tau) \qquad \forall \tau \in V \end{cases} \tag{3.9}$$

under the additional constraints $b(\tau, v) = G(v)$ for all $v \in W$.

Since V and W are Hilbert spaces, and thanks to proposition (3.1), this problem is equivalent to finding $\sigma \in V$ such that:

$$J(\sigma) = min\left\{J(\tau), \tau \in V \text{ and } b(\tau, v) = \int_K f.vdx, \quad \forall v \in W\right\}.$$

Where:

$$J(\tau) = \frac{1}{2}a(\tau, \tau) - l(\tau) \quad \forall \tau \in V$$

We introduce now the following Lagrangian:

$$L(\tau, v) = J(\tau) + b(\tau, v) - G(v)$$

then

$$\left\langle \frac{\partial L}{\partial \tau}(\sigma, u), \tau \right\rangle_V = a(\sigma, \tau) + b(\tau, u) - l(\tau) = 0$$

and

$$\left\langle \frac{\partial L}{\partial v}(\sigma, u), v \right\rangle_W = b(\sigma, v) - G(v) = 0.$$

$\langle ., . \rangle_V$ and $\langle ., . \rangle_W$ are respectively the inner product in V and W.

So the weak formulation of the modified problem (3.6) is given by the following mixed problem [10]:

$$\begin{cases} \text{find}(\sigma, u) \in V \times W \text{verifying :} \\ a(\sigma, \tau) + b(\tau, u) = l(\tau) \qquad \forall \tau \in V \\ b(\tau, v) = G(v) \qquad \forall u \in W \end{cases} \tag{3.10}$$

3.2 Existence and uniqueness of solution

As like as the section (2.3), we deal to demonstrate the existence and uniqueness of solution for the proposed elasticity model under constraint of deformation. We will use for this Brezzi's theorem [10],[12].

Theorem 3.1. *If the bilinear form a(.,.) is symmetric, continuous and coercive on $V \times V$, and the bilinear form b(.,.) is continuous on $V \times W$, then if there exist a real $\gamma > 0$ satisfying the following condition "$inf sup$":*

$$\sup_{\tau \in V}\left(\inf_{v \in W} \frac{b(\tau, v)}{\|v\|_W \|\tau\|_V} \right) \geq \gamma$$

the problem (3.10) admits unique solution (ψ, u).

Proof. Thanks to proposition 3.1, we know that the bilinear form a(.,.) is symmetric and continuous on $V \times V$, and that bilinear form b(.,.) is continuous on $V \times W$. Moreover a(.,.) is coercive. On the other hand, if we choose $\tau = \varepsilon(v)$ for given $v \in W$, then

$$\frac{b(\tau, v)}{\|v\|_W \|\tau\|_V} = \frac{\int_\Omega \tau \times \varepsilon(v) dx}{\|v\|_W \|\tau\|_V} = \frac{\int_\Omega \varepsilon(v) \times \varepsilon(v) dx}{\|v\|_W \|\tau\|_V} = \frac{\|\varepsilon(v)\|_V^2}{\|v\|_W \|\varepsilon(v)\|_V} = 1$$

and

$$\sup_{\tau \in V}\left(\inf_{v \in W} \frac{b(\tau, v)}{\|v\|_W \|\tau\|_V} \right) \geq 1.$$

We deduce that the Elasticity problem (3.1) under constraint of strain admits unique solution. □

3.3 The a priori error estimate

This part is devoted to establish an a priori error estimate for solution to the problem (3.1). For doing that we use the results proven in [10] since the constraint is linear which allows us to express it in form of a kernel of a linear operator B.

Taking advantage of saddle point formulation, the error estimate for the variable u is handled thanks to some properties satisfied by the kernel of B and the kernel of its projection into the finite element space.

Define $M = W$ and $X = V$, and consider the two Hilbert spaces $X_h \subset X$ and $M_h \subset M$ given by the finite element approximation. The approached problem of (3.10) can be written as follows:

$$\begin{cases} Find\ (\sigma_h, u_h) \in X_h \times M_h \text{ which satisfy :} \\ a(\psi_h, \tau_h) + b(\sigma_h, v_h) = l(\tau_h) \qquad \forall \tau_h \in X_h \\ b(\tau_h, v_h) = G(v_h) \qquad \forall v_h \in M_h \end{cases} \tag{3.11}$$

where $a(.,.)$, $b(.,.)$, $l(.)$ and $G(.)$ are defined in the section (3.1).

The norms considered respectively on X_h and M_h are those introduced respectively on X and M.

Theorem 3.2. *If the bi-linear form a(.,.) is continuous and coercive on $X_h \times X_h$, and if there exist a real $\gamma_h > 0$ satisfying the following condition "infsup":*

$$\inf_{v_h \in M_h} \big(\sup_{\tau_h \in X_h} \frac{b(\tau_h, v_h)}{\|\tau_h\|_{X_h} \|v_h\|_{M_h}} \big) \geq \gamma_h \tag{3.12}$$

then the problem (3.11) is well posed.

Proof. $X_h \subset X$ and $M_h \subset M$,Thanks to proposition 3.1, then the bilinear form a(.,.) is coersive and continuous on $X_h \times X_h$ and b(.,.) is a bilinear and continuous form in $X_h \times M_h$. We choose $\tau_h = \varepsilon(v_h)$ for given $v_h \in M_h$, then

$$\frac{b(\tau_h, v_h)}{\|\tau_h\|_{X_h} \|v_h\|_{M_h}} = \frac{\displaystyle\int_\Omega \tau_h \times \varepsilon(v_h) dx}{\|v_h\|_{M_h} \|\tau_h\|_{X_h}} = \frac{\displaystyle\int_\Omega \varepsilon(v_h) \times \varepsilon(v_h) dx}{\|v_h\|_{M_h} \|\tau_h\|_{X_h}} = \frac{\|\varepsilon(v_h)\|^2_{X_h}}{\|v_h\|_{M_h} \|\varepsilon(v_h)\|_{X_h}} = 1 \qquad \forall v_h \in M_h$$

and there exists $\gamma_h = 1$ such that

$$\inf_{v_h \in M_h} \big(\sup_{\tau_h \in X_h} \frac{b(\tau_h, v_h)}{\|\tau_h\|_{X_h} \|v_h\|_{M_h}} \big) \geq \gamma_h. \tag{3.13}$$

Thus the problem (3.11) is well posed. □

Let $B : X \to M'$ the operator introduced by b(.,.) on X and defined by the relation:

$$< B\tau, v >= b(\tau, v)$$

and

$$N(B) = \{\tau \in V / \ b(\tau, v) = 0 \ , \ \ \forall v \in M\}$$

the Kernel of B.

We denote by $B_h : X_h \to M'_h$ the operator introduced by b(.,.) on X_h and defined by the relation:

$$< B_h\tau_h, v_h >= b(\tau_h, v_h)$$

and

$$N(B_h) = \{\tau_h \in X_h / \ b(\tau_h, v_h) = 0 \ , \ \ \forall v_h \in M_h\}$$

the Kernel of B_h [10].

The bi-linear form a(.,.) is continuous and coercive on $X_h \times X_h$, then we have

$$a(\sigma_h, \sigma_h) = \|\sigma_h\|^2_{X_h} \ , \ , \ \ \forall \sigma_h \in N(B_h)$$

and

$$\frac{a(\sigma_h, \sigma_h)}{\|\sigma_h\|^2_{X_h}} = 1, \ , \ \ \forall \sigma_h \in N(B_h). \tag{3.14}$$

Let $\sigma_h \in N(B_h)$, so

$$\sup_{\tau_h \in X_h} \frac{a(\sigma_h, \tau_h)}{\|\tau_h\|_{X_h} \|\sigma_h\|_{X_h}} \big) \geq 1 \tag{3.15}$$

and one can write that:

$$\exists \alpha_h > 0 \ / \qquad \inf_{\sigma_h \in N(B_h)} \big(\sup_{\tau_h \in N(B_h)} \frac{a(\sigma_h, \tau_h)}{\|\tau_h\|_{X_h} \|\sigma_h\|_{X_h}} \big) \geq \alpha_h \tag{3.16}$$

In the following we denote by (σ, u) the solution of the problem (3.10) given by the theorem 3.1 and by (σ_h, u_h) the solution of the discretized problem (3.10) given by the theorem 3.2 thus one can announce the following error estimates.

Lemma 3.3. *Under the hypothesis (3.12) and (3.16), the solution* (σ_h, u_h) *of the problem (3.11) satisfies the following estimates:*

$$||\sigma - \sigma_h||_X \leq C_{1h} \inf_{\tau_h \in X_h}(||\sigma - \tau_h||_X) + C_{2h} \inf_{v_h \in M_h}(||u - v_h||_M)$$
$$||u - u_h||_M \leq C_{3h} \inf_{\tau_h \in X_h}(||\sigma - \tau_h||_X) + C_{2h} \inf_{v_h \in M_h}(||u - v_h||_M)$$

with

$C_{2h} = \dfrac{||b||}{\alpha_h}$ *if* $N(B_h) \not\subseteq N(B)$ *and* $C_{2h} = 0$ *if* $N(B_h) \subset N(B)$,

moreover

$C_{1h} = (1 + \dfrac{||a||}{\alpha_h})(1 + \dfrac{||b||}{\gamma_h})$, $C_{3h} = C_{1h}\dfrac{||a||}{\gamma_h}$ *and* $C_{4h} = (1 + \dfrac{||b||}{\gamma_h}) + C_{2h}\dfrac{||a||}{\gamma_h}$.
α_h *is the coercivity constant of* $a(.,.)$ *and* γ_h *is the infsup condition constant.*

Proof. See [10]. □

Proposition 3.2. *Under the assumptions of theorems 3.1 and 3.2, and supposing that there is a interpolation constant* $c_i > 0$ *which verifies:*

$$\begin{cases} \forall h \in I\!R_+^*, \forall(\tau, v) \in X \times M, \\ \inf\limits_{(\tau_h, v_h) \in (X_h \times M_h)}(||\tau - \tau_h||_X + ||v - v_h||_M) \leq c_i h(||\tau||_X + ||v||_M) \end{cases} \tag{3.17}$$

then the error estimate for the finite element method applied to the problem (3.10) is given by:

$$||\sigma - \sigma_h||_X \leq C_1 h(||\sigma||_X + ||u||_M)$$

and

$$||u - u_h||_M \leq C_2 h(||\sigma||_X + ||u||_M).$$

Proof. Thanks to (3.3) lemma, we have

$$||\sigma - \sigma_h||_X \leq C_{1h} \inf_{\tau_h \in X_h}(||\sigma - \tau_h||_X) + C_{2h} \inf_{v_h \in M_h}(||u - v_h||_M)$$

and:

$$||u - u_h||_M \leq C_{3h} \inf_{\tau_h \in X_h}(||\sigma - \tau_h||_X) + C_{4h} \inf_{v_h \in M_h}(||u - v_h||_M).$$

Let $C1_{max} = max(C_{1h}, C_{2h})$ and $C2_{max} = max(C_{3h}, C_{4h})$ So:

$$||\sigma - \sigma_h||_X \leq C1_{max} \inf_{(\tau_h, v_h) \in (X_h \times M_h)}(||\sigma - \tau_h||_X + ||u - v_h||_M) \tag{3.18}$$

and:

$$||u - u_h||_M \leq C2_{max} \inf_{(\tau_h, v_h) \in (X_h \times M_h)}(||\sigma - \tau_h||_X + ||u - v_h||_M). \tag{3.19}$$

Taking $\tau = \sigma$ and $v = u$ in (3.17), one can conclude that

$$\begin{cases} \forall h \in I\!R_+^*, (\sigma, u) \in X \times M, \\ \inf\limits_{(\tau_h, v_h) \in (X_h \times M_h)}(||\sigma - \tau_h||_X + ||u - v_h||_M) \leq c_i h(||\sigma||_X + ||u||_M) \end{cases} \tag{3.20}$$

We replace (3.20) in (3.18) (3.19), then there exists $c_i > 0$ that:

$$||\sigma - \sigma_h||_X \leq C1_{max}[c_i h(||\sigma||_X + ||u||_M)]$$

and:

$$||u - u_h||_M \leq C2_{max}[c_i h(||\sigma||_X + ||u||_M)].$$

So:

$$||\sigma - \sigma_h||_X \leq C_1 h(||\sigma||_X + ||u||_M)$$

and:

$$||u - u_h||_M \leq C_2 h(||\sigma||_X + ||u||_M).$$

with $C_1 = C1_{max}c_i$ and $C_2 = C2_{max}c_i$.

□

Remark 3.3. This error estimate is also verified for the Poisson problem under constraint.

4 Numerical Results

We used the Free Fem software for numerical simulations for the two presented models: Poisson and elastic one subject to constraint. A simple geometry, a unit square in $2D$ and a unit cube in $3D$ have been choosen for Ω [9],[19],[20].

We give in what follows a graphic representation of the mesh and Level lines of the deformation according to the boundary condition and constraint choice.

4.1 The Poisson model under constraint simulations

The following simulations are performed with zero Dirichlet boundary conditions for two different orientations, $w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Moreover the deformation follow the y-direction.

Comment: In the Fig. 1, the chosen orientation w follows the deformation direction y-axis, then we have a great deformation which follows the y-direction. In the second figure, the orientation w follows the x-axis, so we observe that we have a low deformation.

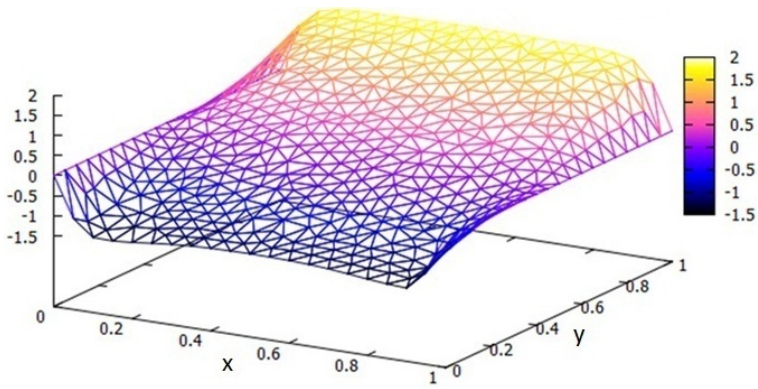

Fig. 1. Poisson model under constraint, with $\nabla uw = 0$ and $w = (0; 1)$

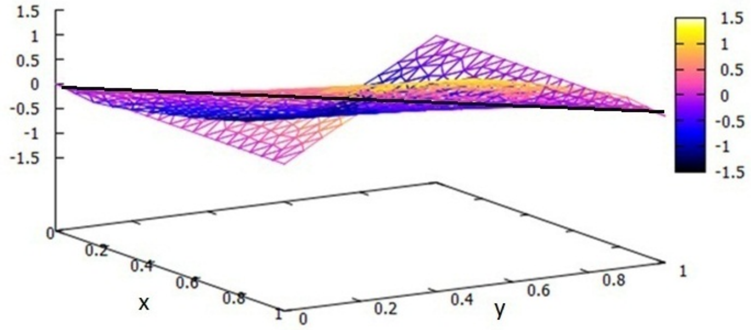

Fig. 2. Poisson model under constraint, with $\nabla uw = 0$ and $w = (1; 0)$

We conclude that when the chosen orientation w follows the deformation direction, we have a great deformation.

4.2 The elasticity model under constraint simulations: 2D dimension

In the case of the elasticity model, simulations are performed under mixed boundary conditions for two different orientations, $w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

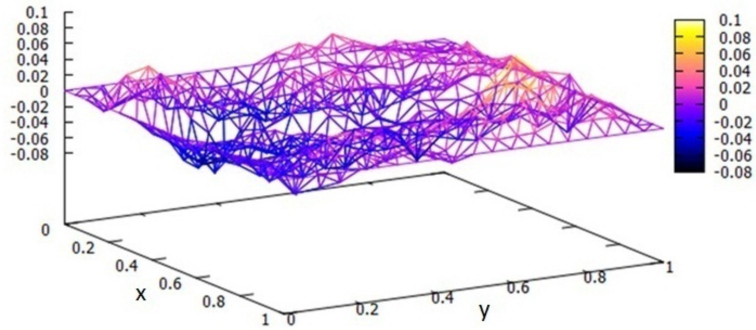

Fig. 3. Elasticity model under constraint $\varepsilon(u)w = 3w$ with $w = (1; 0)$

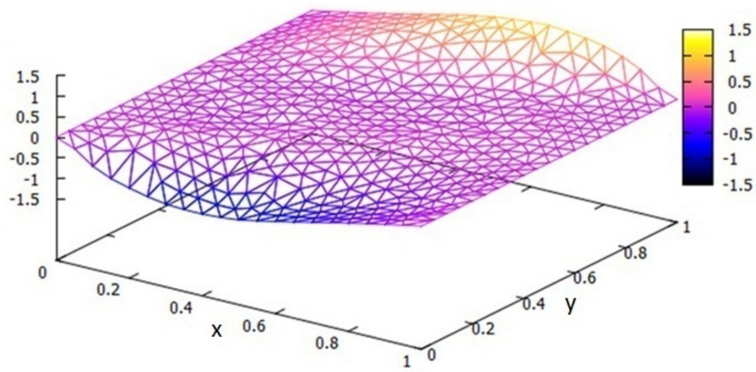

Fig. 4. Elasticity model under constraint $\varepsilon(u)w = 3w$ with $w = (0; 1)$

Comment: As like as the proposed Poisson model, in the Fig. 4, the chosen orientation w follows the deformation direction y-axis, then we have a great deformation which follows the y-direction. In the Fig. 3, the orientation w follows the x-axis, so we observe that we have a low deformation.

We conclude that when the chosen orientation w follows the deformation direction, we have a great deformation.

4.3 The elasticity model under constraint simulations: 3D dimension

We present in this section the simulations performed for the elasticity model under constraint taking into account mixed boundary conditions for different orientations. The chosen deformation direction follows the y-axis, then follows the xy-plane.

Comment: In the Fig. 5, the chosen orientation w follows the deformation direction xy-plane, specially the y-direction, so we remark that we have a great deformation, more then the deformation recorded in the Fig. 6 because the chosen orientation w follows the deformation direction x-axis in xy-plane. When the orientation w follows z-axis, we observe a low deformation. Moreover, when the orientation w follows x-axis, y-axis and z-axis in the same time, we remark a low deformation. We conclude that when the orientation follows the deformation direction.

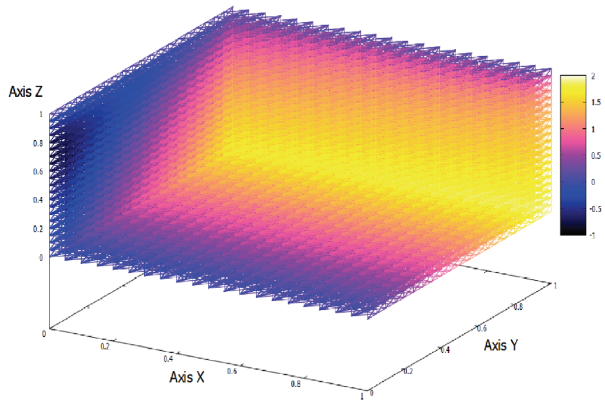

Fig. 5. Elasticity model under constraint $\varepsilon(u)w = 3w$ with $w = (0; 1; 0)$

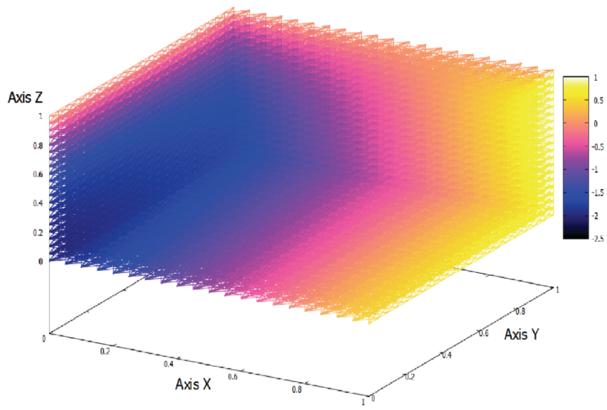

Fig. 6. Elasticity model under constraint $\varepsilon(u)w = 3w$ with $w = (1; 0; 0)$

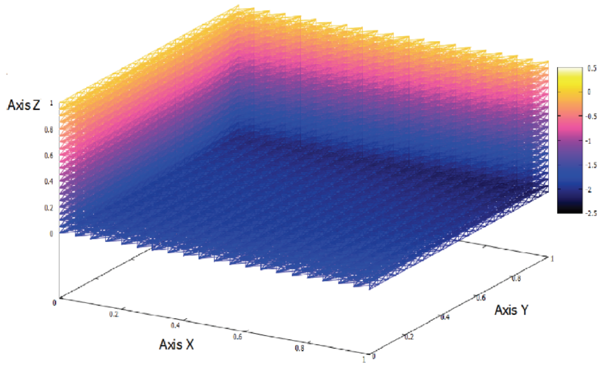

Fig. 7. Elasticity model under constraint $\varepsilon(u)w = 3w$ with $w = (0; 0; 1)$

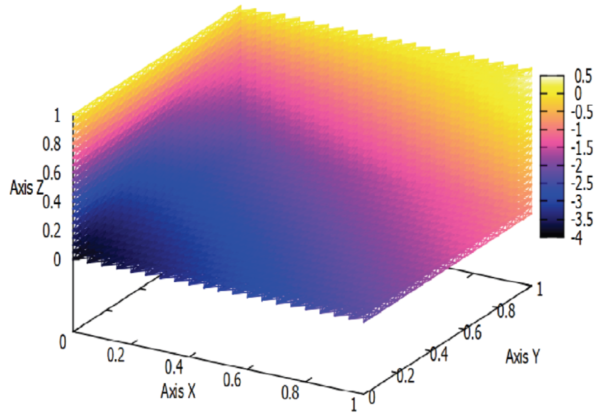

Fig. 8. Elasticity model under constraint $\varepsilon(u)w = 3w$ with $w = (1; 1; 1)$

5 Conclusions

As a conclusion, this work has been devoted to study an elasticity problem under deformation constraint. In first section, we started by studying the Poisson model under constraint for more understand the saddle point problems [10]. The second section was dedicated to study the elasticity problem under deformation constraint. We demonstrate the existence and uniqueness of solution for this problem, and we make an error estimation of solution.

To illustrate numerically what we did in the two section before, we integrated some numerical simulations in the third section.

The results of this study will be useful for proper extraction of characteristic information in a cardiac image. The numerical results will be applied to a cardiac image to detect the contours of the left ventricle under deformation constraint.

In what follows this project, we are in the phase of studying the case of eigenvalues and his simulations, and after that, we will apply these simulations on an image of the heart, to make a correction of the cardiac images segmentation.

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Competing Interests

Authors have declared that no competing interests exist.

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