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A Remark on Common Fixed Points for Two Self - Mappins in Cone Hexagonal Metric Spaces

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Authors' contributions

This work was carried out in collaboration between both authors. Author AA proposed the main idea of this paper, performed all the steps of proofs and wrote the first draft of the manuscript. Author EH managed the analysis of the research work and literature searches. Both authors read and approved the final manuscript.

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ABSTRACT

In this paper, we prove existence of common fixed points for a pair of self mappings in non-normal cone hexagonal metric spaces. The established results extend and improve recent results obtained by many authors. We give example to elucidate our result.

Keywords: Cone hexagonal metric space; common fixed point; coincidence point; contraction mapping principle; weakly compatible maps.

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1 INTRODUCTION

Fixed point theory is one of the traditional theories in mathematics and has a large number of applications in it and many branches of nonlinear analysis. It is well known that the Banach contraction principle [1] is a main result in the fixed point theory, which has been used and extended in many different directions. The study of existence and uniqueness of fixed points of a mapping and common fixed points of two or more mappings has become a subject of great interest. Many authors proved the Banach contraction principle in various generalized metric spaces (e.g., see [2, 3, 4, 5, 6]).

Jungck [7] proved a common fixed point theorem for commuting mappings as a generalization of the Banach's fixed point theorem. The concept of the commutativity has been generalized in several ways. For instance; Sessa [8] introduced the concept of weakly commuting mappings, Jungck [9] extended this concept to compatible maps. In 1998, Jungck and Rhoades [10] introduced the notion of weak compatibility and showed that compatible maps are weakly compatible but the converse need not to be true (e.g., see [11]).

Huang and Zhang [6] introduced the concept of a cone metric space, they replaced the set of real numbers by an ordered Banach space and proved some fixed point theorems for contractive type conditions in cone metric spaces. Later on many authors have (for e.g., [12, 13, 14]) proved some fixed point theorems for different contractive types conditions in cone metric spaces.

Azam et al. [2] introduced the notion of cone rectangular metric space and proved Banach contraction mapping principle in a normal cone rectangular metric space setting. In 2012, Rashwan and Saleh [15] extended and improved the result of Azam et al. [2] by omitting the assumption of normality condition.

Recently, Garg and Agarwal [4] introduced the notion of cone pentagonal metric space and proved Banach contraction mapping principle in a normal cone pentagonal metric space setting.

In 2014, Garg [5] introduced the notion of cone hexagonal metric space and proved Banach contraction mapping principle in a normal cone hexagonal metric space setting.

In the paper [16], Khamsi claims that most of the cone fixed point results are merely copies of the classical ones and that any extension of known fixed point results to cone metric spaces is redundant; also that underlying Banach space and the associated cone subset are not necessary. In fact, Khamsi's approach includes a small class of results and is very limited since it requires only normal cone metric spaces, so that all results with non-normal cones (which are proper extensions of the corresponding results for metric spaces) cannot be dealt with by his approach (for more details, see [17] and the references therein).

Very recently, Auwalu [18] studied common fixed point of a self mapping in cone pentagonal metric space and proved Banach fixed point theory in a cone pentagonal metric space setting by removing the normality condition of the paper [4].

Motivated and inspired by the results of [5, 15, 19], it is our purpose in this paper to continue the study of common fixed points of a pair of self mappings in non-normal cone pentagonal metric space setting. Our results extend and improve the results of [2, 4, 5, 15, 18, 19, 20], and others.

2 PRELIMINARIES

We present some definitions and Lemmas introduced in [2, 4, 5, 6, 12, 15, 20], which will be needed in the sequel.

Definition 2.1. Let E be a real Banach space and P subset of E. P is called a cone if and only if:

(1) P is closed, nonempty and $P \neq \{0\}$;

(2) $a, b \in \mathbb{R}, \quad a, b \geq 0$ and $x, y \in P \Longrightarrow ax + by \in P$;

(3) $x \in P$ and $-x \in P \Longrightarrow x = 0$.

Given a cone $P \subseteq E$, we defined a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$, where $int(P)$ denotes the interior of P.

Definition 2.2. A cone P is called normal if there is a number $k \geq 1$ such that for all $x, y \in E$, the inequality

$$0 \leq x \leq y \Longrightarrow \|x\| \leq k\|y\|. \qquad (2.1)$$

The least positive number k satisfying (2.1) is called the normal constant of P.

In this paper, we always suppose that E is a real Banach space and P is a cone in E with $int(P) \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.

Definition 2.3. Let X be a nonempty set. Suppose the mapping $\rho : X \times X \to E$ satisfies:

(1) $0 < \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$;

(2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;

(3) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$.

Then ρ is called a cone metric on X and (X, ρ) is called a cone metric space.

Remark 2.1. The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, \infty)$ (e.g., see [6]).

Definition 2.4. Let X be a nonempty set. Suppose the mapping $\rho : X \times X \to E$ satisfies:

(1) $0 < \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$;

(2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;

(3) $\rho(x, y) \leq \rho(x, w) + \rho(w, z) + \rho(z, y)$ for all $x, y, z \in X$ and for all distinct points $w, z \in X - \{x, y\}$ [rectangular property].

Then ρ is called a cone rectangular metric on X and (X, ρ) is called a cone rectangular metric space.

Remark 2.2. Every cone metric space is cone rectangular metric space. The converse is not necessarily true (e.g., see [2]).

Definition 2.5. Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies:

(1) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for $x, y \in X$;

(3) $d(x, y) \leq d(x, z) + d(z, w) + d(w, u) + d(u, y)$ for all $x, y, z, w, u \in X$ and for all distinct points $z, w, u, \in X - \{x, y\}$ [pentagonal property].

Then d is called a cone pentagonal metric on X and (X, d) is called a cone pentagonal metric space.

Remark 2.3. Every cone rectangular metric space and so cone metric space is cone pentagonal metric space. The converse is not necessarily true (e.g., see [4]).

Definition 2.6. Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies:

(1) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for $x, y \in X$;

(3) $d(x, y) \leq d(x, z) + d(z, w) + d(w, u) + d(u, v) + d(v, y)$ for all $x, y, z, w, u, v \in X$ and for all distinct points $z, w, u, v \in X - \{x, y\}$ [hexagonal property].

Then d is called a cone hexagonal metric on X and (X, d) is called a cone hexagonal metric space.

Remark 2.4. Every cone pentagonal metric space and so cone rectangular metric space is cone hexagonal metric space. The converse is not true (e.g., see [5]).

Definition 2.7. Let (X, d) be a cone hexagonal metric space. Let $\{x_n\}$ be a sequence in (X, d) and $x \in X$. If for every $c \in E$ with $0 \ll c$ there exist $n_0 \in \mathbb{N}$ and that for all $n > n_0$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Definition 2.8. Let (X, d) be a cone hexagonal metric space. Let $\{x_n\}$ be a sequence in (X, d) and $x \in X$. If for every $c \in E$, with $0 \ll c$ there exist $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called Cauchy sequence in (X, d).

Definition 2.9. Let (X, d) be a cone hexagonal metric space. If every Cauchy sequence is convergent in (X, d), then (X, d) is called a complete cone pentagonal metric space.

Definition 2.10. Let P be a cone defined as above and let Φ be the set of non decreasing continuous functions $\varphi : P \to P$ satisfying:

1. $0 < \varphi(t) < t$ for all $t \in P \setminus \{0\}$,
2. the series $\sum_{n\geq 0} \varphi^n(t)$ converge for all $t \in P \setminus \{0\}$.

From (1), we have $\varphi(0) = 0$ and from (2), we have $\lim_{n\to 0} \varphi^n(t) = 0$ for all $t \in P \setminus \{0\}$.

Definition 2.11. Let T and S be self maps of a nonempty set X. If $w = Tx = Sx$ for some $x \in X$, then x is called a coincidence point of T and S and w is called a point of coincidence of T and S.

Definition 2.12. Two self mappings T and S are said to be weakly compatible if they commute at their coincidence points, that is, $Tx = Sx$ implies that $TSx = STx$.

Lemma 2.1. *Let T and S be weakly compatible self mappings of nonempty set X. If T and S have a unique point of coincidence $w = Tx = Sx$, then w is the unique common fixed point of T and S.*

Lemma 2.2. *Let (X, d) be a cone metric space with cone P not necessary to be normal. Then for all $a, c, u, v, w \in E$, we have*

(1) If $a \leq ha$ and $h \in [0, 1)$, then $a = 0$.

(2) If $0 \leq u \ll c$ for each $0 \ll c$, then $u = 0$.

(3) If $u \leq v$ and $v \ll w$, then $u \ll w$.

Lemma 2.3. *Let (X, d) be a complete cone hexagonal metric space. Let $\{x_n\}$ be a Cauchy sequence in X and suppose that there is natural number N such that:*

1. $x_n \neq x_m$ for all $n, m > N$;

2. x_n, x are distinct points in X for all $n > N$;

3. x_n, y are distinct points in X for all $n > N$;

4. $x_n \to x$ and $x_n \to y$ as $n \to \infty$.

Then $x = y$.

3 MAIN RESULTS

In this section, we derive the main results of our work, which is an extension of Banach contraction principle in cone hexagonal metric space to a pair of two self - mappings. We give an example to illustrate the result.

Theorem 3.1. *Let (X, d) be a cone hexagonal metric space. Suppose the mappings $S, f : X \to X$ satisfy the contractive condition:*

$$d(Sx, Sy) \leq \varphi\big(d(fx, fy)\big), \tag{3.1}$$

for all $x, y \in X$, where $\varphi \in \Phi$. Suppose that $S(X) \subseteq f(X)$ and $f(X)$ or $S(X)$ is a complete subspace of X, then the mappings S and f have a unique point of coincidence in X. Moreover, if S and f are weakly compatible then S and f have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Since $S(X) \subseteq f(X)$, we can choose $x_1 \in X$ such that $Sx_0 = fx_1$. Continuing this process, having chosen x_n in X, we obtain x_{n+1} such that

$$Sx_n = fx_{n+1}, \text{ for all } n = 0, 1, 2, \ldots.$$

We assume that $Sx_n \neq Sx_{n-1}$ for all $n \in \mathbb{N}$. Then from (3.1), it follows that

$$\begin{aligned} d(Sx_n, Sx_{n+1}) &\leq \varphi\big(d(fx_n, fx_{n+1})\big) = \varphi\big(d(Sx_{n-1}, Sx_n)\big) \\ &\leq \varphi^2\big(d(fx_{n-1}, fx_n)\big) \\ &\ \vdots \\ &\leq \varphi^n\big(d(Sx_0, Sx_1)\big). \end{aligned} \tag{3.2}$$

In similar way, it again follows that

$$d(Sx_n, Sx_{n+2}) \leq \varphi^n\big(d(Sx_0, Sx_2)\big), \tag{3.3}$$

$$d(Sx_n, Sx_{n+3}) \leq \varphi^n\big(d(Sx_0, Sx_3)\big), \tag{3.4}$$

$$d(Sx_n, Sx_{n+4}) \leq \varphi^n\big(d(Sx_0, Sx_4)\big). \tag{3.5}$$

Similarly for $k = 1, 2, 3, \ldots,$ it further follows that

$$d(Sx_n, Sx_{n+4k+1}) \leq \varphi^n\big(d(Sx_0, Sx_{4k+1})\big), \tag{3.6}$$

$$d(Sx_n, Sx_{n+4k+2}) \leq \varphi^n\big(d(Sx_0, Sx_{4k+2})\big), \tag{3.7}$$

$$d(Sx_n, Sx_{n+4k+3}) \leq \varphi^n\big(d(Sx_0, Sx_{4k+3})\big), \tag{3.8}$$

$$d(Sx_n, Sx_{n+4k+4}) \leq \varphi^n\big(d(Sx_0, Sx_{4k+4})\big). \tag{3.9}$$

By hexagonal property and (3.2), we have

$$\begin{aligned} d(Sx_0, Sx_5) &\leq d(Sx_0, Sx_1) + d(Sx_1, Sx_2) + d(Sx_2, Sx_3) + d(Sx_3, Sx_4) + d(Sx_4, Sx_5) \\ &\leq d(Sx_0, Sx_1) + \varphi\big(d(Sx_0, Sx_1)\big) + \varphi^2\big(d(Sx_0, Sx_1)\big) + \varphi^3\big(d(Sx_0, Sx_1)\big) + \varphi^4\big(d(Sx_0, Sx_1)\big) \\ &\leq \sum_{i=0}^{4} \varphi^i\big(d(Sx_0, Sx_1)\big), \end{aligned}$$

and

$$\begin{aligned} d(Sx_0, Sx_9) \leq\ & d(Sx_0, Sx_1) + d(Sx_1, Sx_2) + d(Sx_2, Sx_3) + d(Sx_3, Sx_4) + d(Sx_4, Sx_5) \\ & + d(Sx_5, Sx_6) + d(Sx_6, Sx_7) + d(Sx_7, Sx_8) + d(Sx_8, Sx_9) \\ \leq\ & \sum_{i=0}^{8} \varphi^i\big(d(Sx_0, Sx_1)\big). \end{aligned}$$

Now by induction, we obtain for each $k = 1, 2, 3, \ldots$

$$d(Sx_0, Sx_{4k+1}) \leq \sum_{i=0}^{4k} \varphi^i\big(d(Sx_0, Sx_1)\big). \tag{3.10}$$

Also using (3.2), (3.3) and hexagonal property, we have that

$$d(Sx_0, Sx_6) \leq \sum_{i=0}^{3} \varphi^i\big(d(Sx_0, Sx_1)\big) + \varphi^4\big(d(Sx_0, Sx_2)\big),$$

and

$$d(Sx_0, Sx_{10}) \leq \sum_{i=0}^{7} \varphi^i\big(d(Sx_0, Sx_1)\big) + \varphi^8\big(d(Sx_0, Sx_2)\big).$$

By induction, we obtain for each $k = 1, 2, 3, \ldots$

$$d(Sx_0, Sx_{4k+2}) \leq \sum_{i=0}^{4k-1} \varphi^i\big(d(Sx_0, Sx_1)\big) + \varphi^{4k}\big(d(Sx_0, Sx_2)\big). \tag{3.11}$$

Again using (3.2), (3.4) and hexagonal property, we have that

$$d(Sx_0, Sx_7) \leq \sum_{i=0}^{3} \varphi^i\big(d(Sx_0, Sx_1)\big) + \varphi^4\big(d(Sx_0, Sx_3)\big),$$

and

$$d(Sx_0, Sx_{11}) \leq \sum_{i=0}^{7} \varphi^i\big(d(Sx_0, Sx_1)\big) + \varphi^8\big(d(Sx_0, Sx_3)\big).$$

By induction, we obtain for each $k = 1, 2, 3, \ldots$

$$d(Sx_0, Sx_{4k+3}) \leq \sum_{i=0}^{4k-1} \varphi^i\big(d(Sx_0, Sx_1)\big) + \varphi^{4k}\big(d(Sx_0, Sx_3)\big). \tag{3.12}$$

Again using (3.2), (3.5) and hexagonal property, we have that

$$d(Sx_0, Sx_8) \leq \sum_{i=0}^{3} \varphi^i\big(d(Sx_0, Sx_1)\big) + \varphi^4\big(d(Sx_0, Sx_3)\big),$$

and

$$d(Sx_0, Sx_{12}) \leq \sum_{i=0}^{7} \varphi^i\big(d(Sx_0, Sx_1)\big) + \varphi^8\big(d(Sx_0, Sx_3)\big).$$

By induction, we obtain for each $k = 1, 2, 3, \ldots$

$$d(Sx_0, Sx_{4k+4}) \leq \sum_{i=0}^{4k-1} \varphi^i\big(d(Sx_0, Sx_1)\big) + \varphi^{4k}\big(d(Sx_0, Sx_3)\big). \tag{3.13}$$

Using (3.6) and (3.10), for $k = 1, 2, 3, \ldots$, we have

$$\begin{aligned} d(Sx_n, Sx_{n+4k+1}) &\leq \varphi^n \sum_{i=0}^{4k} \varphi^i\big(d(Sx_0, Sx_1)\big) \\ &\leq \varphi^n \Big[\sum_{i=0}^{4k} \varphi^i\big(d(Sx_0, Sx_1) + d(Sx_0, Sx_2) + d(Sx_0, Sx_3) + d(Sx_0, Sx_4)\big)\Big] \\ &\leq \varphi^n \Big[\sum_{i=0}^{\infty} \varphi^i\big(d(Sx_0, Sx_1) + d(Sx_0, Sx_2) + d(Sx_0, Sx_3) + d(Sx_0, Sx_4)\big)\Big]. \end{aligned} \tag{3.14}$$

Similarly for $k = 1, 2, 3, \ldots$, (3.7) and (3.11) implies that

$$\begin{aligned} d(Sx_n, Sx_{n+4k+2}) &\leq \varphi^n \Big[\sum_{i=0}^{4k-1} \varphi^i\big(d(Sx_0, Sx_1)\big) + \varphi^{4k}\big(d(Sx_0, Sx_2)\big)\Big] \\ &\leq \varphi^n \Big[\sum_{i=0}^{\infty} \varphi^i\big(d(Sx_0, Sx_1) + d(Sx_0, Sx_2) + d(Sx_0, Sx_3) + d(Sx_0, Sx_4)\big)\Big]. \end{aligned} \tag{3.15}$$

Again, for $k = 1, 2, 3, \ldots$, (3.8) and (3.12) implies that

$$d(Sx_n, Sx_{n+4k+3}) \leq \varphi^n \Big[\sum_{i=0}^{\infty} \varphi^i\big(d(Sx_0, Sx_1) + d(Sx_0, Sx_2) + d(Sx_0, Sx_3) + d(Sx_0, Sx_4)\big)\Big]. \tag{3.16}$$

Again, for $k = 1, 2, 3, \ldots$, (3.9) and (3.13) implies that

$$d(Sx_n, Sx_{n+4k+4}) \leq \varphi^n \Big[\sum_{i=0}^{\infty} \varphi^i\big(d(Sx_0, Sx_1) + d(Sx_0, Sx_2) + d(Sx_0, Sx_3) + d(Sx_0, Sx_4)\big)\Big]. \tag{3.17}$$

Thus by (3.14), (3.15), (3.16) and (3.17), we have, for each m,

$$d(Sx_n, Sx_{n+m}) \leq \varphi^n \Big[\sum_{i=0}^{\infty} \varphi^i\big(d(Sx_0, Sx_1) + d(Sx_0, Sx_2) + d(Sx_0, Sx_3)\big)\Big]. \tag{3.18}$$

Since $\sum_{i=0}^{\infty} \varphi^i\big(d(Sx_0, Sx_1)+d(Sx_0, Sx_2)+d(Sx_0, Sx_3)+d(Sx_0, Sx_4)\big)$ converges, where $\big(d(Sx_0, Sx_1)+ d(Sx_0, Sx_2) + d(Sx_0, Sx_3) + d(Sx_0, Sx_4)\big) \in P \setminus \{0\}$ and P is closed, then $\sum_{i=0}^{\infty} \varphi^i\big(d(Sx_0, Sx_1) + d(Sx_0, Sx_2) + d(Sx_0, Sx_3) + d(Sx_0, Sx_4)\big) \in P \setminus \{0\}$. Hence

$$\lim_{n\to\infty} \varphi^n \Big[\sum_{i=0}^{\infty} \varphi^i\big(d(Sx_0, Sx_1) + d(Sx_0, Sx_2) + d(Sx_0, Sx_3) + d(Sx_0, Sx_4)\big)\Big] = 0.$$

Then for given $c \gg 0$, there is a natural number N_1 such that

$$\varphi^n\Big[\sum_{i=0}^{\infty}\varphi^i\big(d(Sx_0,Sx_1)+d(Sx_0,Sx_2)+d(Sx_0,Sx_3)+d(Sx_0,Sx_4)\big)\Big] \ll c, \ \ \forall n \geq N_1. \tag{3.19}$$

Thus from (3.18) and (3.19), we have

$$d(Sx_n, Sx_{n+m}) \ll c, \text{ for all } n \geq N_1.$$

Therefore, $\{Sx_n\}$ is a Cauchy sequence in X. Suppose $S(X)$ is a complete subspace of X, then there exists a point $z \in S(X)$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} fx_{n+1} = z$. Also, we can find a point $y \in X$ such that $fy = z$.

We show that $Sy = z$. Given $c \gg 0$, we choose a natural numbers N_2, N_3 such that $d(z, fx_n) \ll \frac{c}{5}, \ \forall n \geq N_2$ and $d(Sx_n, Sx_{n-1}) \ll \frac{c}{5}, \ \forall n \geq N_3$. Since $x_n \neq x_m$ for $n \neq m$, by hexagonal property, we have that

$$\begin{aligned} d(Sy,z) &\leq d(Sy,Sx_n)+d(Sx_n,fx_n)+d(fx_n,fx_{n-1})+d(fx_{n-1},fx_{n-2})+d(fx_{n-2},z)\\ &\leq \varphi\big(d(fy,fx_n)\big)+d(Sx_n,Sx_{n-1})+d(Sx_{n-1},Sx_{n-2})+d(Sx_{n-2},Sx_{n-3})+d(fx_{n-2},z)\\ &< d(z,fx_n)+d(Sx_n,Sx_{n-1})+d(Sx_{n-1},Sx_{n-2})+d(Sx_{n-2},Sx_{n-3})+d(fx_{n-2},z)\\ &\ll \frac{c}{5}+\frac{c}{5}+\frac{c}{5}+\frac{c}{5}+\frac{c}{5} = c, \text{ for all } n \geq N, \end{aligned}$$

where $N = \max\{N_2, N_3\}$. Since c is arbitrary, we have $d(Sy,z) \ll \frac{c}{m}, \ \forall m \in \mathbb{N}$. Since $\frac{c}{m} \to 0$ as $m \to \infty$, we conclude $\frac{c}{m} - d(Sy,z) \to -d(Sy,z)$ as $m \to \infty$. Since P is closed, $-d(Sy,z) \in P$. Hence $d(Sy,z) \in P \cap -P$. By definition of cone we get that $d(Sy,z) = 0$ and so $Sy = fy = z$. Hence, z is a point of coincidence of S and f.

Next, we show that z is unique. For suppose z' be another point of coincidence of S and f, that is $Sx = fx = z'$, for some $x \in X$, then

$$d(z,z') = d(Sy,Sx) \leq \varphi\big(d(fy,fx)\big) = \varphi\big(d(z,z')\big) < d(z,z').$$

Hence $z = z'$. Since S and f are weakly compatible, by Lemma 2.16, z is the unique common fixed point of S and f. This completes the proof of the theorem. ☐

Theorem 3.2. *Let (X,d) be a cone hexagonal metric space. Suppose the mappings $S,f : X \to X$ satisfy the contractive condition:*

$$d(Sfx, Sfy) \leq \varphi\big(d(Sx,Sy)\big), \tag{3.20}$$

for all $x,y \in X$, where $\varphi \in \Phi$. Suppose that S is one to one, $S(X)$ is a complete subspace of X, then the mappings f have a unique fixed point in X. Moreover, if S and f are commuting at the fixed point of f, then S and f have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Define a sequence $\{x_n\}$ in X such that

$$x_{n+1} = fx_n, \text{ for all } n = 0,1,2,\ldots.$$

We assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Then, from (3.20), it follows that

$$\begin{aligned} d(Sx_n,Sx_{n+1}) &= d(Sfx_{n-1},Sfx_n)\\ &\leq \varphi\big(d(Sx_{n-1},Sx_n)\big)\\ &\leq \varphi^2\big(d(Sx_{n-2},Sx_{n-1})\big)\\ &\ \ \vdots\\ &\leq \varphi^n\big(d(Sx_0,Sx_1)\big). \end{aligned} \tag{3.21}$$

In similar way, it again follows that

$$d(Sx_n, Sx_{n+2}) \leq \varphi^n\big(d(Sx_0, Sx_2)\big), \tag{3.22}$$

$$d(Sx_n, Sx_{n+3}) \leq \varphi^n\big(d(Sx_0, Sx_3)\big), \tag{3.23}$$

and

$$d(Sx_n, Sx_{n+4}) \leq \varphi^n\big(d(Sx_0, Sx_4)\big). \tag{3.24}$$

Similarly for $k = 1, 2, 3, \ldots$, it further follows that

$$d(Sx_n, Sx_{n+4k+1}) \leq \varphi^n\big(d(Sx_0, Sx_{4k+1})\big), \tag{3.25}$$

$$d(Sx_n, Sx_{n+4k+2}) \leq \varphi^n\big(d(Sx_0, Sx_{4k+2})\big), \tag{3.26}$$

$$d(Sx_n, Sx_{n+4k+3}) \leq \varphi^n\big(d(Sx_0, Sx_{4k+3})\big), \tag{3.27}$$

$$d(Sx_n, Sx_{n+4k+4}) \leq \varphi^n\big(d(Sx_0, Sx_{4k+4})\big). \tag{3.28}$$

Using the same argument in the proof of Theorem 3.1, we can show that $\{Sx_n\}$ is a Cauchy sequence in X.

Since $S(X)$ is a complete subspace of X, then there exists a point $z \in S(X)$ such that $\lim_{n\to\infty} Sx_{n+1} = \lim_{n\to\infty} Sfx_n = z$. Also, we can find a point $y \in X$ such that $Sy = z$.

We show that $Sfy = Sy$. Given $c \gg 0$, we choose a natural numbers M_1, M_2 such that $d(z, Sx_n) \ll \frac{c}{5},\ \forall n \geq M_1$ and $d(Sx_n, Sx_{n+1}) \ll \frac{c}{5},\ \forall n \geq M_2$. Since $x_n \neq x_m$ for $n \neq m$, by pentagonal property, we have that

$$\begin{aligned}
d(Sy, Sfy) &\leq d(Sy, Sx_n) + d(Sx_n, Sfx_n) + d(Sfx_n, Sfx_{n+1}) + d(Sfx_{n+1}, Sfx_{n+2}) + d(Sfx_{n+2}, Sfy)\\
&\leq d(z, Sx_n) + d(Sx_n, Sx_{n+1}) + \varphi\big(d(Sx_n, Sx_{n+1})\big) + \varphi\big(d(Sx_{n+1}, Sx_{n+2})\big) + \varphi\big(d(Sx_{n+2}, Sy)\big)\\
&< d(z, Sx_n) + d(Sx_n, Sx_{n+1}) + d(Sx_n, Sx_{n+1}) + d(Sx_{n+1}, Sx_{n+2}) + d(Sx_{n+2}, z)\\
&\ll \frac{c}{5} + \frac{c}{5} + \frac{c}{5} + \frac{c}{5} + \frac{c}{5} = c, \text{ for all } n \geq M,
\end{aligned}$$

where $M = \max\{M_1, M_2\}$. Since c is arbitrary, we have $d(Sy, Sfy) = 0$. Therefore, $Sy = Sfy = z$. Since S is one to one $y = fy$. Hence, y is a fixed point of f.

Next, we show that y is unique. For suppose y' be another fixed point of f, that is $fy' = y'$, then

$$d(Sy, Sy') = d(Sfy, Sfy') \leq \varphi\big(d(Sy, Sy')\big) < d(Sy, Sy').$$

Hence $Sy = Sy'$. Since S is one to one, we conclude that $y = y'$.

Since S and f are commuting at the fixed point of f, $Sfy = fSy = Sy$. Therefore Sy is a fixed point of f. Since f has a unique fixed point, we have $Sy = y$. Hence $Sy = fy = y$. This completes the proof of the theorem. □

Example 3.3. *Let* $X = \{1, 2, 3, 4, 5, 6\}$, $E = \mathbb{R}^2$ *and* $P = \{(x, y) : x, y \geq 0\}$ *is a cone in* E. *Define* $d : X \times X \to E$ *as follows:*

$$\begin{aligned}
d(x, x) &= 0, \forall x \in X;\\
d(1, 2) &= d(2, 1) = (5, 10);\\
d(1, 3) = d(3, 1) = d(1, 4) = d(4, 1) &= d(1, 5) = d(5, 1) = d(2, 3) = (3, 2) = d(2, 4) = d(4, 2)\\
&= d(2, 5) = d(5, 2) = d(3, 4) = d(4, 3) = d(3, 5) = d(5, 3) = d(4, 5) = d(5, 4) = (1, 2);\\
d(1, 6) = d(6, 1) = d(2, 6) &= d(6, 2) = d(3, 6) = d(6, 3) = d(4, 6) = d(6, 4) = d(5, 6) = d(6, 5) = (4, 8).
\end{aligned}$$

Then (X, d) is a complete cone hexagonal metric space, but (X, d) is not a complete cone pentagonal metric space because it lacks the pentagonal property:

$$\begin{aligned}(5,10) = d(1,2) &> d(1,3) + d(3,4) + d(4,5) + d(5,2)\\ &= (1,2) + (1,2) + (1,2) + (1,2)\\ &= (4,8), \text{ as } (5,10) - (4,8) = (1,2) \in P.\end{aligned}$$

Now, we define a mapping $S, f : X \to X$ as follows

$$S(x) = \begin{cases} 5, & \text{if } x \neq 6;\\ 2, & \text{if } x = 6.\end{cases}$$

$$f(x) = \begin{cases} 3, & \text{if } x = 1;\\ 1, & \text{if } x = 2;\\ 2, & \text{if } x = 3;\\ 4, & \text{if } x = 4;\\ 5, & \text{if } x = 5;\\ 6, & \text{if } x = 6.\end{cases}$$

Clearly $S(X) \subseteq f(X)$, $f(X)$ is a complete subspace of X and the pairs (S, f) is weakly compatible. The inequality (3.1) *holds for all $x, y \in X$, where $\varphi(t) = \frac{1}{4}t$, and 5 is the unique common fixed point of the mappings S and f.*

4 CONCLUSIONS

Now as corollaries, we recover, extend and generalize the recent results of [2, 4, 5, 15, 18, 19, 20], and others in the literature, to a more general cone hexagonal metric space.

Corollary 4.1. *Let (X, d) be a cone hexagonal metric space. Suppose the mappings $S, f : X \to X$ satisfy the contractive condition:*

$$d(Sx, Sy) \leq \lambda d(fx, fy),$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Suppose that $S(X) \subseteq f(X)$ and $f(X)$ or $S(X)$ is a complete subspace of X, then the mappings S and f have a unique point of coincidence in X. Moreover, if S and f are weakly compatible then S and f have a unique common fixed point in X.

Proof. Define $\varphi : P \to P$ by $\varphi(t) = \lambda t$. Then it is clear that φ satisfies the conditions in definition 2.10. Hence the results follows from Theorem 3.1. □

Corollary 4.2. (see [19]) *Let (X, d) be a cone pentagonal metric space. Suppose the mappings $S, f : X \to X$ satisfy the contractive condition:*

$$d(Sx, Sy) \leq \varphi\big(d(fx, fy)\big),$$

for all $x, y \in X$, where $\varphi \in \Phi$. Suppose that $S(X) \subseteq f(X)$ and $f(X)$ or $S(X)$ is a complete subspace of X, then the mappings S and f have a unique point of coincidence in X. Moreover, if S and f are weakly compatible then S and f have a unique common fixed point in X.

Proof. This follows from the Remark 2.4 and Theorem 3.1. □

Corollary 4.3. (see [15]) *Let (X, d) be a cone rectangular metric space. Suppose the mappings $S, f : X \to X$ satisfy the contractive condition:*

$$d(Sx, Sy) \leq \varphi\big(d(fx, fy)\big),$$

for all $x, y \in X$, where $\varphi \in \Phi$. Suppose that $S(X) \subseteq f(X)$ and $f(X)$ or $S(X)$ is a complete subspace of X, then the mappings S and f have a unique point of coincidence in X. Moreover, if S and f are weakly compatible then S and f have a unique common fixed point in X.

Proof. This follows from the Remark 2.3 and Corollary 4.2. □

Corollary 4.4. (see [20]) *Let (X, d) be a cone hexagonal metric space. Suppose the mapping $S : X \to X$ satisfy the following:*

$$d(Sx, Sy) \leq \varphi\big(d(x, y)\big),$$

for all $x, y \in X$, where $\varphi \in \Phi$. Then S has a unique fixed point in X.

Proof. Putting $f = I$ in Theorem 3.1, where I is the identity mapping. This completes the proof. □

Corollary 4.5. (see [18]) *Let (X, d) be a cone pentagonal metric space. Suppose the mapping $S : X \to X$ satisfy the following:*

$$d(Sx, Sy) \leq \varphi\big(d(x, y)\big),$$

for all $x, y \in X$, where $\varphi \in \Phi$. Then S has a unique fixed point in X.

Proof. This follows from the Remark 2.4 and Corollary 4.4. □

Corollary 4.6. (see [15]) *Let (X, d) be a cone rectangular metric space. Suppose the mapping $S : X \to X$ satisfy the following:*

$$d(Sx, Sy) \leq \varphi\big(d(x, y)\big),$$

for all $x, y \in X$, where $\varphi \in \Phi$. Then S has a unique fixed point in X.

Proof. This follows from the Remark 2.3 and Corollary 4.5. □

Corollary 4.7. (see [5]) *Let (X, d) be a cone hexagonal metric space and P be a normal cone with normal constant k. Suppose the mapping $S : X \to X$ satisfy the following:*

$$d(Sx, Sy) \leq \lambda\big(d(x, y)\big),$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Then S has a unique fixed point in X.

Proof. Putting $f = I$ in Corollary 4.1, where I is the identity mapping. This completes the proof. □

Corollary 4.8. (see [4]) *Let (X, d) be a cone pentagonal metric space and P be a normal cone with normal constant k. Suppose the mapping $S : X \to X$ satisfy the contractive condition:*

$$d(Sx, Sy) \leq \lambda d(x, y),$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Then S has a unique fixed point in X.

Proof. This follows from the Remark 2.4 and Corollary 4.7. □

Corollary 4.9. (see [2]) *Let (X, d) be a cone rectangular metric space and P be a normal cone with normal constant k. Suppose the mapping $S : X \to X$ satisfy:*

$$d(Sx, Sy) \leq \lambda d(x, y),$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Then S has a unique fixed point in X.

Proof. This follows from the Remark 2.3 and Corollary 4.8. □

Corollary 4.10. (see [19]) *Let (X, d) be a cone pentagonal metric space. Suppose the mappings $S, f : X \to X$ satisfy the contractive condition:*

$$d(Sfx, Sfy) \leq \varphi\big(d(Sx, Sy)\big),$$

for all $x, y \in X$, where $\varphi \in \Phi$. Suppose that S is one to one, $S(X)$ is a complete subspace of X, then the mappings f have a unique fixed point in X. Moreover, if S and f are commuting at the fixed point of f, then S and f have a unique common fixed point in X.

Proof. This follows from the Remark 2.4 and Theorem 3.2. □

Corollary 4.11. (see [15]) *Let (X, d) be a cone rectangular metric space. Suppose the mappings $S, f : X \to X$ satisfy the contractive condition:*

$$d(Sfx, Sfy) \leq \varphi\big(d(Sx, Sy)\big),$$

for all $x, y \in X$, where $\varphi \in \Phi$. Suppose that S is one to one, $S(X)$ is a complete subspace of X, then the mappings f have a unique fixed point in X. Moreover, if S and f are commuting at the fixed point of f, then S and f have a unique common fixed point in X.

Proof. This follows from the Remark 2.3 and Corollary 4.10. □

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Authors have declared that no competing interests exist.

References

[1] Banach S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundamenta Mathematicae. 1922;3:133-181.

[2] Azam A, Arshad M, Beg I. Banach contraction principle on cone rectangular metric spaces. Appl. Anal. Discrete Math., 2009;3:236-241.

[3] Branciari A. A fixed point theorem of Banach-Caccippoli type on a class of generalized metric spaces. Publ. Math. Debrecen. 2000;57:31-37.

[4] Garg M, Agarwal S. Banach contraction principle on cone pentagonal metric space. J. Adv. Stud. Topol. 2012;3(1):12-18.

[5] Garg M. Banach contraction principle on cone hexagonal metric space. Ultra Scientist. 2014;26(1):97-103.

[6] Huang LG, Zhang X. Cone metric spaces and fixed point theorems of contractive mappings. J. Math. Anal. and Appl. 2007;332(2):1468-1476.

[7] Jungck G. Commuting mappings and fixed points. Amer. Math. Month. 1976;73:261-263.

[8] Sessa S. On a weak commutativity condition of mappings in fixed point considerations. Publ. Inst. Math. (Beograd)(N.S.). 1982;32:149-153.

[9] Jungck G. Compatible mappings and common fixed points. Int. J. of Math. and Math. Sci. 1986;9:771-779.

[10] Jungck G, Rhoades BE. Fixed points for set valued functions without continuity. Indian J. Pure Appl. Math. 1998;29:227-238.

[11] Pathak HK. Fixed point theorems for weak compatible multi-valued and single-valued mappings. Acta Math. Hungar. 1995;67:69-78.

[12] Abbas M, Jungck G. Common fixed point results for non commuting mappings without continuity in cone metric spaces. J. Math. Anal. and Appl. 2008;341:416-420.

[13] Hlić D, Rakocevic V. Common fixed points for maps on cone metric space. J. Math. Anal. and Appl. 2008;341:876-882.

[14] Rezapour S, Hamlbarani R. Some notes on paper cone metric spaces and fixed point theorems of contractive mappings. J. Math. Anal. and Appl. 2008;345:719-724.

[15] Rashwan RA, Saleh SM. Some fixed point theorems in cone rectangular metric spaces. Mathematica Aeterna. 2012;2:573-587.

[16] Khamsi MA. Remarks on cone metric spaces and fixed point theorems of contractive mappings. Fixed Point Theory Appl. 2010;7.
Article ID 315398
DOI:10.1115/2010/315398

[17] Radenović S, Kadelburg Z, Radenović S. On cone metric spaces: A survey. Nonlinear Anal.: Theory, Methods & Appl. 2011;74:2591-2601.

[18] Auwalu A. Banach fixed point theorem in a cone pentagonal metric spaces. J. Adv. Stud. Topol. 2016;7(2):60-67.

[19] Auwalu A, Hınçal E. Common fixed points of two maps in cone pentagonal metric spaces. Global J. Pure Appl. Math.
10 pages, Article ID 44848, (in press).

[20] Auwalu A, Hınçal E. A note on banach contraction mapping principle in cone hexagonal metric space. British J. Math. Comput. Sci. 2016;16(1):1-12.

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