

Boundary-valued Equations for Force Term in Non-homogeneous Equations of Statics in the Theory of Elastic Mixture

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Authors' contributions

This work was carried out in collaboration of two authors. Author PJU designed the study, wrote the "Abstract", supervised the mathematical formulations and check the necessary equations. Author EN managed the literature searches and carried out the analysis in conjunction with author PJU. The two authors read and approved the final manuscripts.

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Abstract

This paper is devoted to the effect of force term inclusion in the classical field theory of elastic mixture in non-homogeneous equation of Kolosov-Muskhelishvili formula. The problem of plane elasticity for a doubly connected body with outer and inner boundaries in the form of a regular polygon with a common centre and parallel sides is addressed. The unknown arcs are acted upon by external stress with some prescribed boundary conditions. Here we use the method of complex variable theory to determine the applied force and the stress state of the elastic body. Our results show that the theoretical frame work of the forcing term is consistent with the experimental work in the literature.

Keywords: Elasticity; force term inclusion; stress.

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1 Introduction

The application of the methods of conformal mappings and boundary value problems of analytic functions has proved to be the most effective way of solving boundary value problems of elasticity and plate bending. However, for a simply-connected domain, these methods yield effective results (especially for domains mapped onto the circle by rational functions). However, these methods still remain poorly adapted to the use of multiply-connected domains [1]. Nevertheless, for some practically important classes of doubly-connected domains bounded by polygons including the polygonal domain with a curvilinear 2-gonal hole, we may succeed in constructing effectively (in the analytical form) functions conformally mapping this domain onto the circular ring. [2]. In addition to this, the Kolosov-Muskhelishvili methods make it possible to decompose these problems into two Riemann-Hilbert problems for the circular ring and by solving the latter problem, we can construct the sought complex potentials in the analytical form.

The theories of mixtures in the framework of rational continuum thermodynamics have been developed throughout the sixties and seventies, and subsequent development in various constitutive theories and thermodynamic analysis are too numerous to document [3]. Boundary value problems for a finite domain with a part of its boundary being unknown and the other part a polygonal line were solved in [4]. A similar boundary-value problems of plane elasticity for infinite plates weakened by unknown full-strength holes with normal stresses acting on their boundaries and forces applied at infinity were analysed in [5,6,7].

A mixed problem of elasticity was solved in [8,9] for a convex polygon and for a doubly connected domain with a polygonal boundary. Also, linear and non-linear static boundary-value problems for doubly or multiply connected isotropic and anisotropic elastic bodies (plates and shell) were solved by Maksimyuk and Chernyshenko [10], Liu, I-Shih: discussed the entropy flux of transversely isotropic elastic bodies of homogeneous type [11], while [12,13,14] gave a solution of a non-classical problem of oscillation of two component mixtures. A fundamental solution of the system of differential equations of stationary oscillations of two-temperature elastic mixtures theory was provided by [15].

In this paper, we analyse a problem of plane elasticity for a doubly connected body with outer and inner boundaries in the form of regular polygon with a common centre and parallel sides. The sides of the polygon are exposed to external gravitational force and the boundary conditions of the forcing term are determined in order to ascertain the impact of the force on the isotropic elastic body.

This is achieved by deriving the forcing term from the non-homogeneous equation of statics in the theory of elastic mixture. These forces are analysed under two-dimensional stress function from which the equilibrium and compatibility equations are derived. Using the compatibility equation and the stress-strain relations, we derived the basic equations of elasticity. And finally, we derived the boundary equation of the forcing term, from which a graph was generated to explain the stress state of the isotropic elastic body.

2 Mathematical Formulation

We consider a homogeneous isotropic elastic body occupying a doubly connected domain on the complex plane $Z = x_1 + ix_2$. Its outer and inner boundary are L_0 and L_1 respectively and are regular polygons with a common centre $Z = 0$ with parallel sides. The neighbourhood of the vertices of the inner polygon are equal smooth arcs symmetric about the rays coming from the centre to the vertices as in Fig. 1.

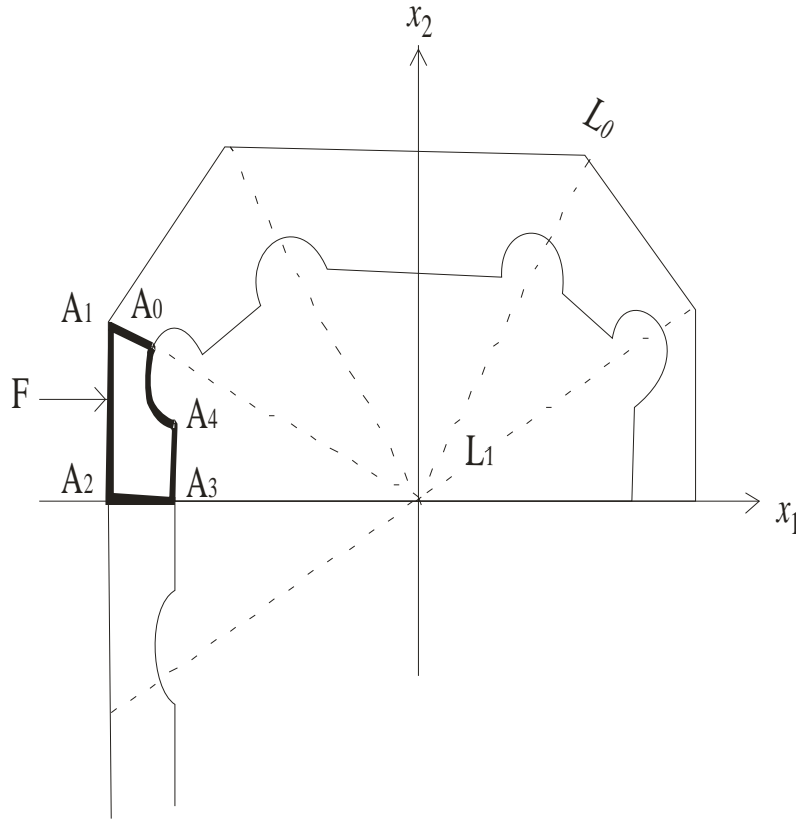


Fig. 1. Isotropic elastic material in a polygonal shape

We assume that this body is inserted into a regular polygonal hole of a rigid body. The boundary of the hole coincided with the outer boundary L_0 of the elastic body. We assume also that the boundary of the hole and the boundary L_0 of the elastic body are smooth, that is, frictional forces are absent. Absolutely smooth rigid punches with rectilinear bases are pressed by a force into all the rectilinear sections of the internal boundary. Under these assumptions, the normal displacements of the outer and inner boundaries are $V_n = 0$ and $V_s = K = \text{constant}$ respectively. The unknown arcs are exposed to external force. The tangential stresses $\tau_s = 0$ on the entire boundary of the domain S .

In our method, we aim to consider the problems of determining the displacement and the stress state of the elastic body, on the assumption that the normal stress σ_s is constant ($\sigma_s = K = \text{constant}$)

3 Method of Solution

We use non-homogeneous equation in the theory of elastic mixtures as our governing equation to derive the forcing term (F) and the general form of [16-18]. It is shown that the displacement vector components are represented in this theory by means of four arbitrary analytic functions. In the two-dimensional case, the basic non-homogeneous governing equation [16] of the theory of elastic mixture has the form:

$$a_1 \Delta u' + b_1 \text{grad div } u' + c \Delta u'' + d \text{grad div } u'' = -\rho_1 F = \psi' \tag{1}$$

$$c \Delta u' + d \text{grad div } u' + a_2 \Delta u'' + b_2 \text{grad div } u'' = -\rho_2 F'' = \psi''$$

where Δ is the two-dimensional Laplacian, grad and div are the principal operators of the field theory, ρ_1 and ρ_2 are the partial densities (positive constants of the mixture), F' and F'' are the mass forces respectively $u' = (u_1', u_2') = w'$ and $u'' = (u_1'', u_2'') = w''$ are the displacement vectors, ψ' and ψ'' denote the product of the partial density ρ and the mass force F respectively, a_1, b_1, c, d, a_2, b_2 are combination of constitutive constants characterising the physical properties of the mixtures [19-20].

3.1 Introducing complex variable theory

We solve equation (1) by introducing the complex variables as follows:

$$z = x_1 + ix_2 \tag{2}_1$$

and

$$\bar{z} = x_1 - ix_2 \tag{2}_2$$

x_1 and x_2 are real and imaginary parts. Thus we obtain

$$2 \left(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right) = 2 \frac{\partial}{\partial z} \quad \text{and} \quad 2 \left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) = 2 \frac{\partial}{\partial \bar{z}} \tag{3}$$

Multiplying the two equations in (3) and simplifying, we have

$$4 \frac{\partial^2}{\partial z \partial \bar{z}} = 4 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) + 4i \left(\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial^2}{\partial x_1 \partial x_2} \right) \tag{4}$$

Let the displacement vectors w' and w'' in its complex form be represented by

$$\left. \begin{aligned} w' &= u_1' + iu_2', & \bar{w}' &= u_1' - iu_2' \\ w'' &= u_1'' + iu_2'', & \bar{w}'' &= u_1'' - iu_2'' \end{aligned} \right\} \tag{5}$$

Substituting equation (5) into equation (4) we have

$$4 \frac{\partial^2 w'}{\partial z \partial \bar{z}} = 4 \left(\frac{\partial w'}{\partial z} + \frac{\partial \bar{w}'}{\partial \bar{z}} \right) - 4i \left(\frac{\partial w'}{\partial z} - \frac{\partial \bar{w}'}{\partial \bar{z}} \right) \tag{6}$$

$$4 \frac{\partial^2 w''}{\partial z \partial \bar{z}} = 4 \left(\frac{\partial w''}{\partial z} + \frac{\partial \bar{w}''}{\partial \bar{z}} \right) - 4i \left(\frac{\partial w''}{\partial z} - \frac{\partial \bar{w}''}{\partial \bar{z}} \right) \tag{7}$$

Note: The displacement vector w' and w'' depends on the elastic and plastic regions (z and \bar{z}).

To make equation (1) solvable, we shall, as in [1], let

$$\Delta u' = 4 \frac{\partial^2 w'}{\partial z \partial \bar{z}} \quad \text{and} \quad \Delta u'' = 4 \frac{\partial^2 w''}{\partial z \partial \bar{z}} \quad (8)$$

and

$$\frac{\partial w'}{\partial x_1} + \frac{\partial \bar{w}'}{\partial \bar{z}} = 2 \left(\frac{\partial u'_1}{\partial x_1} + \frac{\partial u'_2}{\partial x_2} \right) = 4 \operatorname{div} u' = 4\theta'$$

So that,

$$\frac{\partial u'_1}{\partial x_1} + \frac{\partial u'_2}{\partial x_2} = 2 \operatorname{div} u' = 2\theta' \quad (9)$$

Substituting equation (8) and (9), in equation (1) we have

$$\left. \begin{aligned} 4a_1 \frac{\partial^2 w'}{\partial z \partial \bar{z}} + 4c \frac{\partial^2 w''}{\partial z \partial \bar{z}} + 2b_1 \operatorname{grad} \theta' + 2d \operatorname{grad} \theta'' &= \psi' \\ 4c \frac{\partial^2 w'}{\partial z \partial \bar{z}} + 4a_2 \frac{\partial^2 w''}{\partial z \partial \bar{z}} + 2d \operatorname{grad} \theta' + 2b_2 \operatorname{grad} \theta'' &= \psi'' \end{aligned} \right\} \quad (10)$$

Here, our Laplacian is defined as

$$\Delta = \nabla \cdot \nabla = \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial \bar{z}} \quad (11)$$

Substituting for grad in equation (10) with $\frac{\partial}{\partial \bar{z}}$ we have

$$\left. \begin{aligned} 4a_1 \frac{\partial^2 w'}{\partial z \partial \bar{z}} + 4c \frac{\partial^2 w''}{\partial z \partial \bar{z}} + 2b_1 \frac{\partial \theta'}{\partial \bar{z}} + 2d \frac{\partial \theta''}{\partial \bar{z}} &= \psi' \\ 4c \frac{\partial^2 w'}{\partial z \partial \bar{z}} + 4a_2 \frac{\partial^2 w''}{\partial z \partial \bar{z}} + 2d \frac{\partial \theta'}{\partial \bar{z}} + 2b_2 \frac{\partial \theta''}{\partial \bar{z}} &= \psi'' \end{aligned} \right\} \quad (12)$$

Which on factorization gives:

$$\begin{aligned} \frac{\partial}{\partial \bar{z}} \left(4a_1 \frac{\partial w'}{\partial z} + 4c \frac{\partial w''}{\partial z} + 2b_1 \theta' + 2d \theta'' \right) &= \psi' \\ \frac{\partial}{\partial \bar{z}} \left(4c \frac{\partial w'}{\partial z} + 4a_2 \frac{\partial w''}{\partial z} + 2d \theta' + 2b_2 \theta'' \right) &= \psi'' \end{aligned} \quad (13)$$

Under certain assumption from formula the integral in [11], the above equations (13), gives

$$\left. \begin{aligned} 4a_1 \frac{\partial w'}{\partial z} + 4c \frac{\partial w''}{\partial z} + 2b_1 \theta' + 2d \theta'' &= \frac{\partial \psi^*}{\partial \bar{z}} \\ 4c \frac{\partial w''}{\partial z} + 4a_2 \frac{\partial w''}{\partial z} + 2d \theta' + 2b_2 \theta'' &= \frac{\partial \psi^{**}}{\partial \bar{z}} \end{aligned} \right\} \quad (14)$$

where $\frac{\partial \psi^*}{\partial \bar{z}}$ is the analytic non-homogeneous term. defined as

$\psi^* = u + iv$ is the displacement function at the plastic region and $\bar{z} = x_1 - ix_2$, is the complex conjugate function.

3.2 For the non-homogeneous term $\left(\frac{\partial \psi^*}{\partial \bar{z}} \right)$

Equating the non-homogeneous part of equations (14) and (1), we have

$$\frac{\partial \psi^*}{\partial \bar{z}} = \rho F \quad (15)$$

So that

$$\frac{\partial(u + iv)}{\partial(x_1 - ix_2)} = \frac{\partial u}{\partial x_1} - \frac{\partial v}{\partial x_2} + i \frac{\partial u}{\partial x_2} + i \frac{\partial v}{\partial x_1} = \rho F_1 + i \rho F_2 \quad (16)$$

Separating and equating the real and imaginary parts

$$\frac{\partial u}{\partial x_1} - \frac{\partial v}{\partial x_2} = \rho F_1 \quad \text{and} \quad \frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1} = \rho F_2 \quad (17)$$

Introducing the new arbitrary analytic functions ϕ and η , we have from, [11] that

$$u = \frac{\partial \phi}{\partial x_1} + \frac{\partial \eta}{\partial x_2} \quad \text{and} \quad v = -\frac{\partial \phi}{\partial x_2} + \frac{\partial \eta}{\partial x_1} \quad (18)$$

Using equation (18) and substituting for u and v in equation (17), we obtain

$$\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} = \rho F_1 \quad \text{and hence} \quad \nabla^2 \phi = \rho F_1 \quad (19)$$

$$\text{Similarly, } \nabla^2 \eta = \rho F_2 \quad (20)$$

From equation 16, we have;

$$\nabla^2 \varphi + i \nabla^2 \eta = \rho F_1 + i \rho F_2 = \rho F \quad \text{or} \quad \nabla^2 (\varphi + i \eta) = \rho F \quad (21)$$

Note:

The classical field theory describing gravity is the Newtonian gravitation, which describes the gravitational force F , as a mutual-interaction between two masses, M and m , (where M is the isotropic elastic body, Fig. 1 and m is the object of our forcing term). So that

$$F = -\frac{GMm}{r^2} \quad (22)$$

where G and r are earth gravitational constant and distance respectively.

The massive body M has a gravitational field g . Since the gravitational force F , is conservative, the field g , can be written as a gradient of gravitational scalar potential ϕ , that is

$$g = -\nabla \phi \quad (23)$$

Also, in the case of a gravitational field due to an attracting massive objects of density ρ , Gauss' law for gravity in differential form can be used to obtain the corresponding Poisson equation for gravity, [14]. Gauss' law for gravity is

$$\nabla \cdot g = -4\pi G\rho \quad (24)$$

Substituting for $g = -\nabla \phi$ in equation (23)

$$\nabla^2 \phi = 4\pi G\rho \quad (25)$$

Equation (25) is called Poisson equation for gravity [2]

Hence, equation (21) is equivalent to equation (25), because it involves the mutual interaction between the isotropic elastic body M (Fig. 1) and the object of our forcing term (m).

That is

$$\nabla^2 (\varphi + i \eta) = \nabla^2 \phi = \rho F = 4\pi G\rho. \quad ; \quad \rho F = 4\pi G\rho \quad \text{and} \quad F = 4\pi G \quad (26)$$

Hence, our forcing term is a gravitational force, and it is Poisson in nature, as such, it is restricted to a plane.

3.3 Boundary conditions for the forces (f) on the plane

The boundary conditions are obtained from the requirement that at the surface of the body, the stress must equal the imposed surface force [3]. If the surface element dA has a normal \hat{n} then we evaluate the stress vector at this point as shown in [13].

We now specify stresses in different directions

(i). X-direction: $F_x = dA\sigma_x \cos \alpha + dA\tau_{xy} \sin \alpha$ and denoting $\frac{F_x}{dA} = \bar{X}$,

we have,

$$\bar{X} = \sigma_x \cos \alpha + \tau_{xy} \sin \alpha \tag{27}$$

(ii). In Y-direction, similarly $\bar{Y} = \tau_{xy} \sin \alpha + \sigma_y \cos \alpha$ (28)

Hence, force balance on the plane is given in the following matrix as:

$$\bar{X} = \bar{Y} = F = \begin{pmatrix} \sigma_x \cos \alpha & \tau_{xy} \sin \alpha \\ \tau_{xy} \sin \alpha & \sigma_y \cos \alpha \end{pmatrix} \tag{29}$$

Letting, $\sigma_x = a$, $\tau_{xy} = b$ and $\sigma_y = c$.

So that

$$F = \begin{pmatrix} a \cos \alpha & b \sin \alpha \\ b \sin \alpha & c \cos \alpha \end{pmatrix}$$

or

$$|F| = a c \cos^2 \alpha - b^2 \sin^2 \alpha \tag{30}$$

At equilibrium $F = 0$, so that,

$$\frac{ac}{b^2} = \sigma(\alpha) \text{ (general stress) and } \sigma(\alpha) = \tan^2 \alpha \tag{31}$$

Equations (30) and (31) are the main results

4 Results

From [13] the inner displacement vector is given as

$$\bar{V}_n(t) = \frac{(\lambda - 1)k(h - l)}{2\mu} \tag{32}$$

From our main result, the boundary equation is

$$F = \sigma(\cos^2 \alpha - \sin^2 \alpha) \tag{33}$$

At equilibrium $F = 0$. Equation (30) becomes

$$\sigma(\alpha) = \tan^2 \alpha = 1, \quad \alpha = 45^\circ \quad (34)$$

The complete Kolosov-Muskhelishvili formula for a displacement vector [13] is

$$2\mu V_n(t) - \bar{V}_n(t) = F \quad (35_a)$$

$$V_n(t) - \frac{\bar{V}_n(t)}{2\mu} = \frac{F}{2\mu} \quad (35_b)$$

$$V_n(t) - k_1 \bar{V}_n(t) = k^* F. \quad (35_c)$$

$$k_1 = \frac{1}{2\mu}, \quad k^* = (2\mu)^{-1}$$

From the known constant values [13] our graph is shown below.

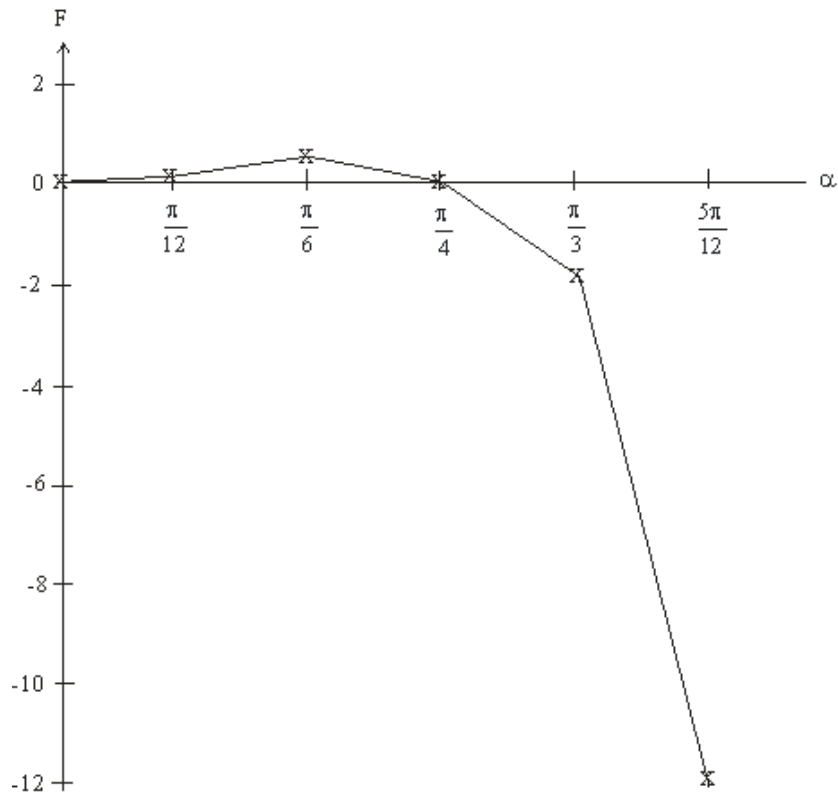


Fig. 2. A graph of angle α against the force F

5 Discussion of Results

The graph in Fig. 2 above results from the relationship between the angle α and the applied external force F . When the mass and momentum resulting from the forces are conserved, then the stability of the system in which the forces act upon, depends on its perfect angular formation. In the graph, from $\frac{\pi}{12}$ to $\frac{\pi}{6}$, there is a slight rise of the curve above the horizontal axis, indicating an increase in the energy level of the particles in the solid. Beyond $\frac{\pi}{6}$, it drop and touch the horizontal axis at $\frac{\pi}{4}$. This implies that, all the forces in that system are resolved to zero at that point and that the stability of the continuum is attain at $\frac{\pi}{4}$. Beyond $\frac{\pi}{4}$, the curve drops below the horizontal axis to the negative vertical axis, indicating that at angle $\frac{\pi}{3}$, $\frac{5\pi}{12}$ etc., the system is unstable.

6 Conclusion

In this paper, we considered the problem of non-homogeneous equation of statics in the theory of elastic mixture, using complex variable theory. It was found that our theoretical solution for the stress state of the isotropic elastic body (Fig. 1) is consistent with the experimental existing result.

Competing Interests

Authors have declared that no competing interests exist.

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