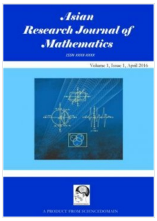

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Solving Fuzzy Heat Equation by Using Numerical Methods

A. Hosseinpour[1*]

[1] *Young Researchers and Elite Club, Kermanshah Branch, Islamic Azad University, Kermanshah, Iran.*

Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

This research proposes an explicit method to solve fuzzy heat equation with integral boundary conditions. The necessary materials and preliminaries are expressed, and a finite difference scheme for one dimensional heat equation is considered. Here, boundary conditions include integral equations which are approximated by the composite trapezoid rule. Finally, an example in order to illustrate the numerical results is given. In this example, the Hausdorff distance between exact solution and approximate solution is obtained.

Keywords: Explicit method; fuzzy numbers; fuzzy heat equation; finite difference scheme.

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1 Introduction

In this paper, the explicit method to solve fuzzy heat equation with nonlocal boundary conditions are explained. The importance of this problem is that the mathematical modeling of many scientific

Corresponding author: E-mail: a_hosseinpour@outlook.com;

phenomena involves nonlocal boundary conditions [1]. Therefore, the numerical solution of parabolic partial differential equations with nonlocal boundary conditions has become an area of interest, in recent years. The increase in research on fuzzy mathematics, necessitate the study of fuzzy partial differential equations and fuzzy derivatives. Prerequisite of this topic is fuzzy derivative which was first studied by Chang and Zadeh in [2]. This discussion was followed by Dubois and Prade by defining the extension principle in [1]. Puri and Relescu in [3] and Goestschel and Voxman in [4] considered some new methods in this context. The boundary value problems for first order fuzzy differential equations have been surveyed by several authors [5, 6, 7, 8, 9]. Afterwards, many authors studied fuzzy partial differential equations [10, 11, 12]. In this direction, Mutlu et al. introduced a more general iteration method to solve initial value problems which can be extended for fuzzy diferential equations [13, 14]. In this work, fuzzy heat equation with nonlocal boundary conditions is studied. In section 2, some preliminaries are presented. In section 3, an explicit method to compute fuzzy heat equation is presented. Here, the integral terms are estimated by the composite trapezoid rule. Finally, in the last section, an example is presented.

2 Materials and Definitions

Let X be a universal set, then the fuzzy set $\widetilde{A}$ in X is determined by $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)) \mid x \in X\}$. The function $\mu_{\widetilde{A}} : X \longrightarrow [0, 1]$ is called the membership function, where $\mu_{\widetilde{A}}(x)$ is the grade of membership of x in $\widetilde{A}$. The range of the membership function is a subset of the nonnegative real numbers whose supremum is finite.

Definition 2.1. [15] Consider a fuzzy set $\widetilde{A}$ in X and any real number α, then the α-cut or α-level set of $\widetilde{A}$ is denoted by A_α and defined as $A_\alpha = \{x \in X \mid \mu_{\widetilde{A}}(x) \geq \alpha\}$. Similarly, $A'_\alpha = \{x \in X \mid \mu_{\widetilde{A}}(x) > \alpha\}$ is called strong α-cut.

Definition 2.2. [16] The triangular fuzzy number $\widetilde{N}$ is defined by three numbers $\alpha < m < \beta$ as $\widetilde{N} = (\alpha, m, \beta)$ and represented as bellow:

$$\mu_{\widetilde{A}}(x) = \begin{cases} \frac{x-\alpha}{m-\alpha} & \alpha \leq x \leq m \\ 1 & x = m \\ \frac{x-m}{\beta-x} & m \leq x \leq \beta \\ 0 & x = 0 \end{cases}$$

If $\alpha > 0$ ($\alpha \geq 0$), then $\widetilde{A} > 0$ ($\widetilde{A} \geq 0$).
If $\beta < 0$ ($\beta \leq 0$), then $\widetilde{A} < 0$ ($\widetilde{A} \leq 0$).

Definition 2.3. [15] An arbitrary fuzzy number is shown, in the parametric form, by an ordered pair of functions $(\underline{a}(r), \overline{a}(r))$ with $0 \leq r \leq 1$ satisfying the following requirements:

1. $\underline{a}(r)$ is a bounded left semicontinuous non-decreasing function over [0,1],
2. $\overline{a}(r)$ is a bounded left semicontinuous non-increasing function over [0,1],
3. $\underline{a}(r) \leq \overline{a}(r)$, $0 \leq r \leq 1$.

Peculiarly, if $\underline{a}$ and $\overline{a}$ are linear functions, then we have a triangular fuzzy number. A crisp number a is represented by $\underline{a} = \overline{a} = a$, for all $0 \leq r \leq 1$.

Definition 2.4. [17] For any fuzzy numbers $u = (\underline{u}(r), \overline{u}(r))$ and $v = (\underline{v}(r), \overline{v}(r))$, the algebraic operations are defined as bellow:

1. $ku = \begin{cases} (k\underline{u}, k\overline{u}) & k \geq 0 \\ (k\overline{u}, k\underline{u}) & k < 0 \end{cases}$
2. $u + v = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$

3. $u - v = (\underline{u}(r) - \overline{v}(r), \overline{u}(r) - \underline{v}(r))$
4. $u \cdot v = (\min S, \max S)$, where $S = \{\underline{uv}, \underline{u}\overline{v}, \overline{u}\underline{v}, \overline{uv}\}$.

Remark 2.1. Since the α-cut of the fuzzy numbers is always a closed and bounded interval, then we can write $\widetilde{A}_\alpha = [\underline{a}(\alpha), \overline{a}(\alpha)]$, for all $\alpha \in R$.

Definition 2.5. [15] Assume that $u = (\underline{u}(r), \overline{u}(r))$ and $v = (\underline{v}(r), \overline{v}(r))$ are two fuzzy numbers. The Hausdorff metric D_H between u and v is defined by:

$$D_H(u, v) = \max_{r \in [0,1]} \{|\underline{u}(r) - \overline{v}(r)|, |\overline{u}(r) - \underline{v}(r)|\}. \tag{2.1}$$

This metric is considered as a bound for computing error, and by using that the difference between exact solution and approximate solution is obtained.

3 Finite Difference Method

In this section, an explicit difference scheme is applied to acquire the numerical solution of fuzzy heat equation:

$$(D_t - a^2 D_x^2)\widetilde{U} = \widetilde{0}, \qquad x \in (0, 1), t \in (0, 1] \tag{3.1}$$

with the following boundary conditions

$$\begin{cases} \widetilde{U}(0, t) = \int_0^1 k_0(x)\widetilde{U}(x, t)dx + \widetilde{g}_0(t) \\ \widetilde{U}(1, t) = \int_0^1 k_1(x)\widetilde{U}(x, t)dx + \widetilde{g}_1(t) \end{cases} \tag{3.2}$$

and the initial condition

$$\widetilde{U}(x, 0) = \widetilde{g}(x), \qquad x \in (0, 1) \tag{3.3}$$

where $\widetilde{g}_0$, $\widetilde{g}_1$ and $\widetilde{g}$ are known fuzzy functions, $\widetilde{U}$ is an unknown fuzzy function which must be determined and k_0 and k_1 are crisp known functions.

Let $\widetilde{U}$ be a fuzzy function of two independent crisp variables x and t. Define $I = \{(x, t)|0 \le x \le 1,\ 0 \le t \le T\}$. An α-cut of $\widetilde{U}(x, t)$ and its parametric form is presented as $\widetilde{U}(x, t)[\alpha] = [\underline{U}(x, t; \alpha), \overline{U}(x, t; \alpha)]$. Assume that the functions $\underline{U}(x, t; \alpha)$ and $\overline{U}(x, t; \alpha)$ have continuous partial differential, then $(D_t - a^2 D_x^2)\overline{U}(x, t; \alpha)$, and $(D_t - a^2 D_x^2)\underline{U}(x, t; \alpha)$ are continuous, for all $(x, t) \in I$, $\alpha \in [0, 1]$. The domain $[0, 1] \times [0, T]$ is partitioned into $M \times N$ sub-domain with spatial step size $h = \frac{1}{N}$ and $k = \frac{T}{M}$ in x-direction and t-direction, respectively. The grid points are given by $x_i = ih$ and $t_j = jk$, for $i = 0, 1, \cdots, N$ and $j = 0, 1, \cdots, M$. The value of $\widetilde{U}$ at the point $p(x_i, t_j)$ is denoted by $\widetilde{U}_p = \widetilde{U}(x_i, t_j) = \widetilde{U}_{i,j}$, and the parametric form of fuzzy number $\widetilde{U}_{i,j}$ is $\widetilde{U}_{i,j} = (\underline{U}_{i,j}, \overline{U}_{i,j})$. Hence, we have:

$$\begin{cases} (D_t)\widetilde{U}_{i,j} = (D_t\underline{U}_{i,j}, D_t\overline{U}_{i,j}) \\ (D_x^2)\widetilde{U}_{i,j} = (D_x^2\underline{U}_{i,j}, D_x^2\overline{U}_{i,j}) \end{cases}$$

In this stage, the Taylor-series expansion is exerted at a given point p to determine the following schemes:

$$\begin{cases} D_x^2\underline{U}_{i,j+1} \simeq \frac{\underline{u}_{i-1,j} - 2\overline{u}_{i,j} + \underline{u}_{i+1,j}}{h^2} \\ D_x^2\overline{U}_{i,j+1} \simeq \frac{\overline{u}_{i-1,j} - 2\underline{u}_{i,j} + \overline{u}_{i+1,j}}{h^2} \end{cases} \tag{3.4}$$

and:

$$\begin{cases} D_t\underline{U}_{i,j} \simeq \frac{\underline{u}_{i,j+1}-\overline{u}_{i,j}}{k} \\ D_t\overline{U}_{i,j} \simeq \frac{\overline{u}_{i,j+1}-\underline{u}_{i,j}}{k} \end{cases} \tag{3.5}$$

Parametric representation of the heat equation (3.1) will be:

$$\begin{cases} D_t\underline{U}_{i,j} - a^2 D_x^2\overline{U}_{i,j} = \widetilde{0} \\ D_t\overline{U}_{i,j} - a^2 D_x^2\underline{U}_{i,j} = \widetilde{0} \end{cases} \tag{3.6}$$

By inserting (3.4) and (3.5) into the Equation (3.6), the difference scheme for fuzzy heat equation is obtained as bellow:

$$\begin{cases} \frac{\underline{u}_{i,j+1}-\overline{u}_{i,j}}{k} - a^2\frac{\overline{u}_{i-1,j}-2\underline{u}_{i,j}+\overline{u}_{i+1,j}}{h^2} = 0 \\ \frac{\overline{u}_{i,j+1}-\underline{u}_{i,j}}{k} - a^2\frac{\underline{u}_{i-1,j}-2\overline{u}_{i,j}+\underline{u}_{i+1,j}}{h^2} = 0 \end{cases} \tag{3.7}$$

By simplifying the above equations, we obtain:

$$\begin{cases} \underline{u}_{i,j+1} = r\overline{u}_{i-1,j} + (1-2r)\underline{u}_{i,j} - r\overline{u}_{i+1,j} \\ \overline{u}_{i,j+1} = r\underline{u}_{i-1,j} + (1-2r)\overline{u}_{i,j} + r\underline{u}_{i+1,j} = \overline{u}_{i,j} \end{cases} \tag{3.8}$$

Where $r = \frac{ka^2}{h^2}$. Let $\widetilde{U} = (\underline{u}, \overline{u})$ be an exact solution of the approximate difference equations. According to Equation (3.8), we have $2(N-1)$ equations with $2(N+1)$ unknowns, therefore we need four more equations. The required equations are obtained by boundary conditions (3.2) which are approximated by the trapezoid rule. Thus, we have

$$a_0\widetilde{U}_{0,j+1} + \sum_{i=1}^{N-1} a_i\widetilde{U}_{i,j+1} + a_N\widetilde{U}_{N,j+1} \approx -\widetilde{g}_{0,i+1}$$

$$b_0\widetilde{U}_{0,j+1} + \sum_{i=1}^{N-1} b_i\widetilde{U}_{i,j+1} + b_N\widetilde{U}_{N,j+1} \approx -\widetilde{g}_{1,i+1}$$

where

$$a_0 = \frac{h}{2}\ k_0(x_0) - 1 \qquad a_N = \frac{h}{2}\ k_0(x_N)$$

$$b_N = \frac{h}{2}\ k_1(x_N) - 1 \qquad b_0 = \frac{h}{2}\ k_1(x_0)$$

and

$$a_i = hk_0(x_i) \quad , \quad b_i = hk_1(x_i) \qquad i = 1, \ldots, N-1$$

The parametric form of fuzzy numbers $\widetilde{g}_0$ and $\widetilde{g}_1$ are as bellow:

$$\widetilde{g}_0 = (\underline{g}_0, \overline{g}_0) \qquad \widetilde{g}_1 = (\underline{g}_1, \overline{g}_1)$$

We obtain

$$\widetilde{u}_{0,j+1} = Y^{-1}[Z_0(hk_1(x_N) - 2) - Z_1 hk_0(x_N)],$$
$$\widetilde{u}_{M,j+1} = Y^{-1}[Z_1(hk_0(x_0) - 2) - Z_0 hk_1(x_0)].$$

Where,

$$Z_0 = -2h \sum_{i=1}^{N-1} k_0(x_i)\widetilde{u}_{i,j+1} - 2\widetilde{g}_{0,j+1} \quad , \quad Z_1 = -2h \sum_{i=1}^{N-1} k_1(x_i)\widetilde{u}_{i,j+1} - 2\widetilde{g}_{1,j+1},$$

and

$$Y = (hk_0(x_0) - 2)(hk_1(x_N) - 2) - h^2 k_1(x_0)k_0(x_N) \neq 0.$$

Equations (3.8) emply that:

$$\widetilde{U}_{i,j+1} = r\widetilde{U}_{i-1,j} + (1-2r)\widetilde{U}_{i,j} + r\widetilde{U}_{i+1,j}, \quad i = 1, \ldots, N-1, j = 0, 1, \ldots, M$$

By this way, the approximate solution of fuzzy heat equation can be found.

4 Numerical Results

In this section, a numerical example is provided to describe how our method works. Let us consider the following fuzzy heat equation:

$$\frac{\partial \widetilde{U}}{\partial t}(x,t) - \frac{1}{8}\frac{\partial^2 \widetilde{U}}{\partial x^2}(x,t) = \exp(t)[4x^2 + t], \quad 0 < x < 1, \ t > 0$$

with the following nonlocal boundary conditions

$$\widetilde{U}(0,t) = \int_0^1 x\widetilde{U}(x,t)dx + \frac{1}{2}(t-2)\exp(t)$$
$$\widetilde{U}(1,t) = \int_0^1 x\widetilde{U}(x,t)dx + \frac{1}{2}(t+6)\exp(t)$$

and the initial condition $\widetilde{U}(x,0) = 4\widetilde{K}x^2$, where, $\widetilde{K}[\alpha] = [\underline{k}(\alpha), \overline{k}(\alpha)] = [\alpha - 1, 1 - \alpha]$. It is easily seen the exact solution for

$$\frac{\partial \underline{U}}{\partial t}(x,t;\alpha) - \frac{1}{8}\frac{\partial^2 \overline{U}}{\partial x^2}(x,t;\alpha) = \exp(t)[4x^2 + t] - \alpha$$
$$\frac{\partial \overline{U}}{\partial t}(x,t;\alpha) - \frac{1}{8}\frac{\partial^2 \underline{U}}{\partial x^2}(x,t;\alpha) = \exp(t)[4x^2 + t] + \alpha$$

is

$$\underline{U}(x,t;\alpha) = \underline{k}(\alpha)\exp(t)[4x^2 + t]$$
$$\overline{U}(x,t;\alpha) = \overline{k}(\alpha)\exp(t)[4x^2 + t]$$

The Tables (1) and (2) show the exact and approximate solutions with spatial step h=0.005 and time step k=0.00001 at x=0.5 and t=0.05, respectively. The Housdroff distances between solutions are shown in each table. We observe that our method is accurate for heat equation to four decimal places. The numerical and exact solutions for this example in place x=0.5 and at time t=0.05 are illustrated in Figs. (1) and (2), respectively.

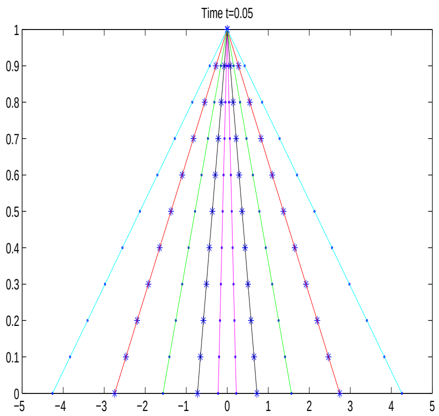

Fig. 1. Exact and computational solution at time $t = 0.05$

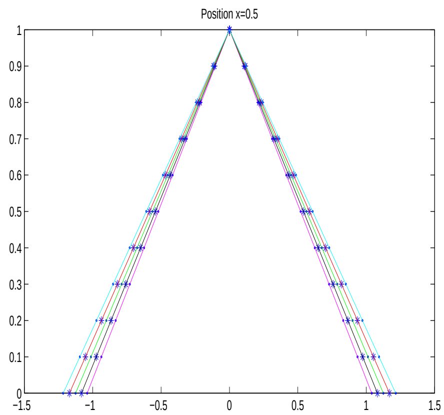

Fig. 2. Exact and computational solution at position $x = 0.5$

Table 1. Error at $t = 0.05$ with $h = 0.005$

x	Exact	Numerical	Error
0.2	0.2208	0.2208	0.0000
0.4	0.7254	0.7254	0.0000
0.6	1.5664	1.5664	0.0000
0.8	2.7438	2.7438	0.0000
1.0	4.2576	4.2576	0.0000

Table 2. Error at $x = 0.25$ with $h = 0.005$

t	Exact	Numerical	Error
0.02	1.0406	1.0406	0.0000
0.04	1.0824	1.0824	0.0000
0.06	1.1255	1.1255	0.0000
0.08	1.1700	1.1700	0.0000
0.10	1.2157	1.2157	0.0000

5 Conclusions

Our purpose in this article is solving fuzzy partial differential equation (FPDE) including integral terms. We presented an explicit method to solve this equation, and finally in the last section, we given an example to consider the numerical results. We also compared the approximate solution and exact solution by Hausdorf distance between them.

Competing Interests

Author has declared that no competing interests exist.

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