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# 3- Point Single Hybrid Block Method (3PSHBM) for Direct Solution of General Second Order Initial Value Problem of Ordinary Differential Equations

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***Authors' contributions***

*This work was carried out in collaboration between both authors. Author BGO designed the study and author EOO wrote the literature review. Author BGO handled the development of the scheme. Author EOO analyzed the basic properties of the method and wrote the programming code for the implementation of the method. Author BGO wrote the first draft of the manuscript. Author EOO put finishing touches to the manuscript and author BGO carried out all the editorial corrections on the final manuscript. Both authors read and approved the final manuscript.*

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## ABSTRACT

In this research article, a Three Point Single Hybrid Block Method for Direct solution of general second order ordinary differential equations is considered. The method was derived by collocation and interpolation of power series approximation to generate a continuous linear multistep method. The resultant method was evaluated at selected grid and off grid points to generate a discrete block method. The basic properties of the method such as order, error constant, zero Stability and consistency are investigated. The new method was tested on some numerical examples which ranges from linear, non-linear and real life problem of initial value problems and was found to be more efficient and give better approximation than the existing methods.

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## 1. INTRODUCTION

In this paper, initial value problem (IVP) of general second order ordinary differential equations (ODEs) of the form:

$$y'' = f(x, y, y'), y(a) = y_0, y' = \eta_0 \quad (1)$$

is considered.

Many problems in the form of (1) may not be easily solved analytically, therefore, an approximate numerical integrations are often developed to solve (1). We often reduce them to systems of first order ordinary differential equations and use appropriate method to solve them. The reduction appraoch has been discussed by several authors such as Fatunla [1] and Lambert [2]. To avoid the rigour of reducing (1) to equivalent system of first order ordinary differential equations, authors proposed linear multistep method to solve equaion (1) directly. Among such authors are Awoyemi [3,4], Awoyemi and Kayode [5], Adesanya et al. [6], Badmus and Yahaya [7]. According to Awoyemi [3], continuous linear multistep method has greater advantages over the discrete method in that they give better error estimation, provide a simplified coefficient for further analytical work at different points, and guarantee easy appropriation of solution at all interior points of the integration interval. In Lambert [2], an optimal two step method called the numerov's method was discussed. Among the authors that proposed continuous linear multistep methods are Awoyemi [4], Onumanyi et al. [8], Adesanya et al. [6], Okunuga [9], to mention a few. These authors individually implemented their methods with predictor-correct, block method and adopted Taylor series expansion to supply starting values. According to Adesanya [10], the setback of the predictor-corrector method is that it is very costly as subroutines are very complicated to write because of the special techniques required to supply starting values and for varying the step size which leads to longer computer time and more human effort. The predictors they developed are not of the same order with the correctors. Hence, it affects the accuracy of the method.

Also, various authors such as Olanegan et al. [11], Bolarinwa et al. [12] developed the hybrid method. And Ali Shokri [13] proposed a symmetric P-Stable hybrid Obrechkoff method for the numerical solution of general second order initial value problems. This hybrid method, while retaining certain charateristics of the continuous linear multistep method, shares with Runge-Kutta methods the property of utilizing data at other points, other than the step point $x_{n+j}, j = 0, 1 \cdots n-1$. This method is useful in reducing the step number of a scheme and the scheme still remain zero stable. Since the predictor-corrector method has not met the requirements above, hence there is a need to develop other method to cater for the draw-backs. Therefore, scholars developed block method to cater for the setback of predictor-corrector method. Among such authors are Awoyemi [4], Jator and Li [14], Majid et al. [15] and Adesanya [16]. Fatunla [17] and Mohammed et al. [18] independently proposed the use of block method as predictors for the numerical solution of second order ordinary differential equations.

Recently, single hybrid three-step and non hybrid four-step block methods for solving third order ordinary differential equations was considered by Ogunware et al. [19].

In this article, we are motivated by the need to develop a new continuous three point single hybrid block method with one-off point for direct solution of general second order ordinary differential equation of initial value problems which can handle general second order ordinary differential equation more accurately and efficiently than the existing methods.

## 2. METHODOLOGY

We consider power series as an approximate solution to the general second order ordinary differential equations initial value problems of the form (1) to be

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j \quad (2)$$

where *aj* are parameters to be determined, r and s are number of interpolation and collocation points.

The first and second derivatives of (2) are obtained as

$$y'(x) = \sum_{j=0}^{r+s-1} j a_j x^{j-1} \quad (3)$$

$$y''(x) = \sum_{j=0}^{r+s-1} j(j-1) a_j x^{j-2} \quad (4)$$

The combination of (4) and (1) gives the differential system below

$$y''(x) = \sum_{j=0}^{r+s-1} j(j-1) a_j x^{j-2} = f(x, y, y') \quad (5)$$

Collocating (5) at $x_{n+j},\ j = 0, \frac{1}{2}, 1, 2, 3$ and interpolating (2) at $x_{n+j}, j = \frac{1}{2}, 1$. Also rewriting them in matrix form as

$$\begin{bmatrix} 1 & x_{n+1/2} & x_{n+1/2}^2 & x_{n+1/2}^3 & x_{n+1/2}^4 & x_{n+1/2}^5 & x_{n+1/2}^6 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 \\ 0 & 0 & 2 & 6x_{n+1/2} & 12x_{n+1/2}^2 & 20x_{n+1/2}^3 & 30x_{n+1/2}^4 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 \\ 0 & 0 & 2 & 6x_{n+2} & 24x_{n+2}^2 & 60x_{n+2}^3 & 120x_{n+2}^4 \\ 0 & 0 & 2 & 6x_{n+3} & 24x_{n+3}^2 & 60x_{n+3}^3 & 120x_{n+3}^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} y_{n+1/2} \\ y_{n+1} \\ f_n \\ f_{n+1/2} \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix} \quad (6)$$

Solving for $a_j, j = 0(1)6$ in (6) using Gaussian elimination method and substituting into (2) gives a linear multistep method with continuous coefficients in the form:

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + h^2 \left( \sum_{j=0}^{k} \beta_j(x) f_{n+j} + \beta_v(x) f_{n+v} \right) \quad (7)$$

Where $y(x)$ is the numerical solution of the initial value problem and $v = \frac{1}{2}$. $\alpha_j$ and $\beta_j$ are continuous coefficients. And that $\alpha_o$ and $\beta_o$ are not both zero since (7) is continuous and differentiable. Hence it is evaluated along with its derivatives at the entire grid and off grid points. This generate a block method for the general second order ordinary differential equation of the form

$$A^{(0)} Y_n = A^{(i)} Y_{n-1} + h^{\mu} [B^{(i)} F_n + B^{(0)} F_{n-1}]$$

Where

$Y_n = [y_{n+1}, y_{n+2}, \ldots, y_{n+r}]^T$, $Y_{n-1} = [y_{n-1}, y_{n-2}, \ldots, y_n]^T$, $F_n = [F_n, f_{n+1,} f_{n+2}, \ldots, f_{n+k}]^T$
$F_{n-1} = [f_{n-1}, f_{n-2,} f_{n-3}, \ldots, f_n]^T$, and $\mu$ is the order of the differential equation.

This gives the independent solution $\{y_{n+j}\},\ \ i = 1(1)k$ without overlapping

$$f_{n+j} = f(x_{n+j,} y_{n+j,} y'_{n+j}) \quad (8)$$

Using the transformation

$$t = \frac{1}{h}(x - x_{n+2}), dt = \frac{1}{h} dx \quad (9)$$

The coefficients of $y_{n+j}$ and $f_{n+j}$ are obtained as:

$$
\begin{aligned}
&\alpha_{1/2}(t) = -2-2t \\
&\alpha_1(t) = 3+2t \\
&\beta_0(t) = \frac{h^2}{5760}\left[171+411t-480t^3-160t^4+144t^5+64t^6\right] \\
&\beta_{\frac{1}{2}}(t) = \frac{-h^2}{900}\left[93+269t-320t^3-80t^4+96t^5+32t^6\right] \\
&\beta_1(t) = \frac{h^2}{1920}\left[1371+2075t-960t^3-80t^4+240t^5+64t^6\right] \\
&\beta_2(t) = \frac{h^2}{5760}\left[669+2381t+2880t^2+1120t^3-320t^4-336t^5-64t^6\right] \\
&\beta_3(t) = \frac{h^2}{28800}\left[-189-477t+960t^3+1040t^4+432t^5+64t^6\right]
\end{aligned}
\tag{10}
$$

Equation (10) is substituted in (7) and evaluated at the non-interpolation points i.e. $t = 1, 0, -2$ , gives

$$y_{n+3} - 5y_{n+1} + 4y_{n+1/2} = \frac{h^2}{960}\left[25f_n - 96f_{n+\frac{1}{2}} + 1355f_{n+1} + 1055f_{n+2} + 61f_{n+3}\right] \tag{11}$$

$$y_{n+2} - 3y_{n+1} + 2y_{n+1/2} = \frac{h^2}{192000}\left[5700f_n - 19840f_{n+\frac{1}{2}} + 137100f_{n+1} + 22300f_{n+2} - 1260f_{n+3}\right] \tag{12}$$

$$y_{n+1} - 2y_{n+1/2} + y_n = \frac{h^2}{19200}\left[390f_n + 4032f_{n+\frac{1}{2}} + 370f_{n+1} + 10f_{n+2} - 2f_{n+3}\right] \tag{13}$$

Also the first derivative of (10) gives:

$$
\begin{aligned}
&\alpha'_{\frac{1}{2}}(t) = \frac{-2}{h} \\
&\alpha'_1(t) = \frac{2}{h} \\
&\beta'_0(t) = \frac{h}{5760}\left[411-1440t^2-640t^3+720t^4+384t^5\right] \\
&\beta'_{1/2}(t) = \frac{-h}{900}\left[269-960t^2-320t^3+480t^4+192t^5\right] \\
&\beta'_1(t) = \frac{h}{1920}\left[2075-2880t^2-320t^3+1200t^4+384t^5\right] \\
&\beta'_2(t) = \frac{h}{5760}\left[2381+5760t+3360t^2-1280t^3-1680t^4-384t^5\right] \\
&\beta'_3(t) = \frac{h}{28800}\left[-477+2880t^2+4160t^3+2160t^4+384t^5\right]
\end{aligned}
\tag{14}
$$

Equation (14) is also substituted in (7) and evaluated at the entire grid and off-grid points i.e at $t=-2,-1,-1/2,0,1$ which gives the following equations:

$$5760hy'_n-11520y_{n+1}-11520y_{n+1/2}=h^2\left[-997f_n-3360f_{n+1/2}+81f_{n+1}-51f_{n+2}+7f_{n+3}\right] \quad (15)$$

$$14400hy'_{n+1/2}-28800y_{n+1}+28800y_{n+1/2}=h^2\left[150f_n-2576f_{n+1/2}-1245f_{n+1}+80f_{n+2}-9f_{n+3}\right] \quad (16)$$

$$28800hy'_{n+1}-57600y_{n+1}+57600y_{n+1/2}=h^2\left[-256f_n+2656f_{n+1/2}+4965f_{n+1}-175f_{n+2}+19f_{n+3}\right] \quad (17)$$

$$28800hy'_{n+2}-57600y_{n+1}+57600y_{n+1/2}=h^2\left[2055f_n-8608f_{n+1/2}+31125f_{n+1}+11905f_{n+2}-477f_{n+3}\right] \quad (18)$$

$$28800hy'_{n+3}-57600y_{n+1}+57600y_{n+1/2}=h^2\left[-2825f_n+10848f_{n+1/2}+6885f_{n+1}+40785f_{n+2}+9107f_{n+3}\right] \quad (19)$$

## 3. MATHEMATICAL FORMULATION OF THE METHOD

Normalizing the combination of equation (11), (12), (13) and (15) give equation (20) below:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1/2}\\y_{n+1}\\y_{n+2}\\y_{n+3}\end{bmatrix}=\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n-1/2}\\y_{n-1}\\y_{n-2}\\y_n\end{bmatrix}+h\begin{bmatrix}0&0&0&1/2\\0&0&0&1\\0&0&0&2\\0&0&0&3\end{bmatrix}\begin{bmatrix}y'_{n-1/2}\\y'_{n-1}\\y'_{n-2}\\y'_n\end{bmatrix}+$$
$$h^2\begin{bmatrix}\frac{49}{600}&\frac{-101}{3840}&\frac{1}{256}&\frac{-29}{57600}\\\frac{28}{75}&\frac{-1}{30}&\frac{1}{120}&\frac{-1}{900}\\\frac{64}{75}&\frac{2}{3}&\frac{2}{15}&\frac{-2}{225}\\\frac{36}{25}&\frac{27}{20}&\frac{9}{8}&\frac{3}{50}\end{bmatrix}\begin{bmatrix}f_{n+1/2}\\f_{n+1}\\f_{n+2}\\f_{n+3}\end{bmatrix}+h^2\begin{bmatrix}0&0&0&\frac{763}{11520}\\0&0&0&\frac{11}{72}\\0&0&0&\frac{16}{45}\\0&0&0&\frac{21}{40}\end{bmatrix}\begin{bmatrix}f_{n-1/2}\\f_{n-1}\\f_{n-2}\\f_n\end{bmatrix} \quad (20)$$

Writing out (20) explicitly, we have

$$y_{n+\frac{1}{2}}=y_n+\left(\frac{1}{2}\right)hy'_n+h^2\left(\frac{763}{11520}f_n+\frac{49}{600}f_{n+1/2}-\frac{101}{3840}f_{n+1}+\frac{1}{256}f_{n+2}-\frac{29}{57600}f_{n+3}\right) \quad (21)$$

$$y_{n+1}=y_n+hy'_n+h^2\left(\frac{11}{72}f_n+\frac{28}{75}f_{n+1/2}-\frac{1}{30}f_{n+1}+\frac{1}{120}f_{n+2}-\frac{1}{900}f_{n+3}\right) \quad (22)$$

$$y_{n+2}=y_n+2hy'_n+h^2\left(\frac{16}{45}f_n+\frac{64}{75}f_{n+1/2}+\frac{2}{3}f_{n+1}+\frac{2}{15}f_{n+2}-\frac{2}{225}f_{n+3}\right) \quad (23)$$

$$y_{n+3}=y_n+3hy'_n+h^2\left(\frac{21}{40}f_n+\frac{36}{25}f_{n+1/2}+\frac{27}{20}f_{n+1}+\frac{9}{8}f_{n+2}+\frac{3}{50}f_{n+3}\right) \quad (24)$$

Substituting (21)-(24) into (16)-(19) gives the following:

$$\begin{aligned}y'_{n+\frac{1}{2}}&=y'_n+\frac{h^2}{28800}\left[5285f_n+11520f_{n+1/2}-2895f_{n+1}+415f_{n+2}-53f_{n+3}\right]\\y'_{n+1}&=y'_n+\frac{h^2}{1800}\left[295f_n+1216f_{n+1/2}+285f_{n+1}+5f_{n+2}-f_{n+3}\right]\\y'_{n+2}&=y'_n+\frac{h^2}{225}\left[55f_n+64f_{n+1/2}+240f_{n+1}+95f_{n+2}+4f_{n+3}\right]\\y'_{n+3}&=y'_n+\frac{h^2}{200}\left[15f_n+192f_{n+1/2}+45f_{n+1}+285f_{n+2}+63f_{n+3}\right]\end{aligned} \quad (25)$$

## 4. ANALYSIS OF THE BLOCK

### 4.1 Order of the Block

In this section, we discuss the estimation of the order and error constant of the block with the difference equation of the form:

$$L\left[y(x),h\right]=\sum_{j=1}^{k}a_j y(x+jh)-h^2\sum_{j=0}^{k}b_j y'''(x+jh) \tag{26}$$

If we assume that $y(x)$ has as many higher derivatives as we require, we can expand the terms in (11) as a Taylor series about the point $x$ to obtain the expansion;

$$L\left[y(x),h\right]=C_0y(x_n)+C_1hy'(x_n)+C_2h^2y''(x_n)+C_3h^3y'''(x_n)+.....+C_ph^py^{(p)}(x_n) \tag{27}$$

Where the constant coefficient $c_q, q=0,1,.......$ are given as follows

$$C_0=\sum_{j=0}^{k}\alpha_j$$

$$C_1=\sum_{j=1}^{k}j\alpha_j$$

.

.

.

$$C_q=\frac{1}{q!}[\sum_{j=1}^{k}j^q\alpha_j-q(q-1)\sum_{j=1}^{k}j^{q-3}\beta_j]$$

Definition: The method (11) are said to be of order p if, in (11) $C_0=C_1=C_2=C_3=.....C_P=C_{P+1}=0$ and $C_{p+2}\neq 0$. Thus $C_{P+2}$ is the error constant.

For our hybrid method, expanding (21)-(24) in Taylor series expansion gives

$$\begin{bmatrix} \sum_q^{\infty}\left(\frac{h^q}{q!}y^q\right)-y_n-\frac{1}{2}hy'_n-\sum_q^{\infty}\left(\frac{h^{q+2}}{q!}y^{q+2}\right)-\left(\frac{49}{600}(\frac{1}{2})^q-\frac{101}{3840}(1)^q+\frac{1}{256}(2)^q-\frac{29}{57600}(3)^q\right) \\ \sum_q^{\infty}\left(\frac{h^q}{q!}y^q\right)-y_n-hy'_n-\sum_q^{\infty}\left(\frac{h^{q+2}}{q!}y^{q+2}\right)-\left(\frac{28}{75}(\frac{1}{2})^q-\frac{1}{30}(1)^q+\frac{1}{120}(2)^q-\frac{1}{900}(3)^q\right) \\ \sum_q^{\infty}\left(\frac{h^q}{q!}y^q\right)-y_n-2hy'_n-\sum_q^{\infty}\left(\frac{h^{q+2}}{q!}y^{q+2}\right)-\left(\frac{64}{75}(\frac{1}{2})^q+\frac{2}{3}(1)^q+\frac{2}{15}(2)^q-\frac{2}{225}(3)^q\right) \\ \sum_q^{\infty}\left(\frac{h^q}{q!}y^q\right)-y_n-3hy'_n-\sum_q^{\infty}\left(\frac{h^{q+2}}{q!}y^{q+2}\right)-\left(\frac{36}{25}(\frac{1}{2})^q+\frac{25}{20}(1)^q+\frac{9}{8}(2)^q-\frac{3}{50}(3)^q\right) \end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Hence the block is of order 5, with error constant of $\left[\frac{37}{6451200},\frac{41}{201600},\frac{313}{12600},\frac{9489}{22400}\right]^T$

### 4.2 Zero Stability of the Block

Definition: The block is said to be zero stable if the roots $z_{s,} s = 1, 2, 3, ..., n$ of the characteristics polynomial $\rho(z)$ defined by

$\rho(z) = \det\left(zA - E\right)$ satisfies $|z_s| \leq 1$ and the roots $|z_s| = 1$ is simple.

For our hybrid method,

$$A = \left[ z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right] = 0$$

$$A = z^4 - z^3 = 0, z = 0, 0, 0, 1$$

Hence the block is zero stable. Lambert [2].

**Theorem 1**: Convergence Lambert [2], Fatunla [17]

The necessary and sufficient condition for a linear multistep method to be convergent is for it to be consistent and zero stable. From the theorem above, our block method is convergent.

## 5. NUMERICAL EXPERIMENTS

Our method is adopted on some initial value problems of general second order ordinary differential equations ranging from linear, non-linear and real life problem to test the accuracy of our method.

**Problem 1**:

We consider the non- liner scalar test equation which was solved by Ali Shokri [13]

$$y'' = -w^2 y, \qquad y(0) = 1, \qquad y'(0) = 0, \ y(0) = 1, \ y'(0) = 0, h = \frac{\pi}{800}, \ \frac{\pi}{1600}, w = 10,$$

with exact solution: $y(x) = \cos(wx)$

**Table 1a. The result of test problem 1 [at $h = \frac{\pi}{800}$]**

| Point | Exact solution | Computed solution | Error in our new method $h = \frac{\pi}{800}$, P=5, K=3 | Error in Ali Shokri [13] $h = \frac{\pi}{800}$, P=6,K=2 |
|---|---|---|---|---|
| 5 $\pi$ | 0.809016994374948 | 0.809016994188417 | 1.86531E-10 | 1.6532E-07 |
| 10 $\pi$ | 0.233445363855907 | 0.233445363717165 | 1.38742E-10 | 3.1128E-07 |
| 15 $\pi$ | -0.271440449865071 | -0.271440449668074 | 1.96997E-10 | 6.8875E-07 |
| 20 $\pi$ | -0.785316930880744 | -0.785316929948751 | 9.31993E-10 | 9.5563E-06 |

**Table 1b. The result of test problem 1 [at $h = \frac{\pi}{1600}$]**

| Point | Exact solution | Computed solution | Error in our new method $h = \frac{\pi}{1600}$, P=5, K=3 | Error in Ali Shokri [13] $h = \frac{\pi}{1600}$,P=6,K=2 |
|---|---|---|---|---|
| 5 $\pi$ | 0.951056516295154 | 0.951056516359792 | 6.4638E-11 | 2.5567E-09 |
| 10 $\pi$ | 0.785316930880745 | 0.785316930909005 | 2.8260E-11 | 5.2398E-09 |
| 15 $\pi$ | 0.603555941953572 | 0.603555941917062 | 3.6510E-11 | 6.9971E-08 |
| 20 $\pi$ | 0.327630179561695 | 0.327630179319564 | 2.4213E-10 | 9.20138E-08 |

**Problem 2:**

Consider a slightly stiff linear second order problem

$$y'' = y', \; y(0) = 1, \; y'(0) = 0, h = 0.1$$

Exact solution: $y(x) = 1 - \exp(x)$

**Table 2. The result of test problem 2**

| X | Exact Solution | Computed Solution | Error in Our New Method k=3, h=0.1,P=5 | Error in Anake [2011] k=1, h=0.1, P=6 |
|---|---|---|---|---|
| 0.1 | -0.10517091808 | -0.1051709180 | 7.56500E-11 | 0.84736252E-07 |
| 0.2 | -0.2214027582 | -0.2214027580 | 1.60170E-10 | 0.11719652E-05 |
| 0.3 | -0.3498588076 | -0.3498588074 | 1.76000E-10 | 0.32170472E-05 |
| 0.4 | -0.4918246976 | -0.4918246983 | 6.07843E-10 | 0.64094269E-05 |
| 0.5 | -0.6487212707 | -0.6487212722 | 1.47289E-09 | 0.10967802E-04 |
| 0.6 | -0.8221188004 | -0.8221188029 | 2.53363E-09 | 0.1744180E-04 |
| 0.7 | -1.0137527074 | -1.0137527122 | 4.78762E-09 | 0.25228466E-04 |
| 0.8 | -1.2255409285 | -1.2255409358 | 7.27701E-09 | 0.35553649E-04 |
| 0.9 | -1.4596031112 | -1.4596031213 | 1.01696E-08 | 0.48501651E-04 |
| 1.0 | -1.7182818285 | -1.7182818433 | 1.48265E-08 | 0.64509947E-04 |

**Problem 3: Real-life Problem**

**Cooling of a Body**

The temperature $y$ degrees of a body, $t$ minutes after being placed in a certain room, satisfies the differential equation $3\frac{d^2y}{dt^2} + \frac{dy}{dt} = 0$. By using the substitution $z = \frac{dy}{dt}$, or the otherwise, find $y$ in terms of $t$ given that $y = 60$ when $t = 0$ and $y = 35$ when $t = 6\ln 4$. Find after how many minutes the rate of cooling of the body will have fallen below one degree per minute, giving your answer correct to the nearest minute.

Formulation of the Problem

$$y'' = \frac{-y'}{3}, \; y(0) = 60, \; y'(0) = -\frac{80}{9}, \; h = 0.1$$

**Analytical Solution**

$$y(x) = \frac{80}{3} e^{-\left(\frac{1}{3}\right)x} + \frac{100}{3}$$

**Table 3. The result of test problem 3 (Real-life Problem)**

| X | Exact-Solution | Computed-Solution | Error in Our New Method K=3, h=0.1, P=5 | Error in Olanegan[2014] K=2, h=0.1, P=5 |
|---|---|---|---|---|
| 0.1 | 59.1257626795201 | 59.1257626795556 | 3.55E-11 | 7.476427E-06 |
| 0.2 | 58.2801862675098 | 58.2801862675556 | 4.58E-11 | 2.939419E-05 |
| 0.3 | 57.4623311476256 | 57.4623311475556 | 7.00E-11 | 6.480165 E-05 |
| 0.4 | 56.6712885078119 | 56.6712885078054 | 6.50E-12 | 1.127905 E-05 |
| 0.5 | 55.9061793304163 | 55.9061793304496 | 3.33E-11 | 1.724976E-04 |
| 0.6 | 55.1661534154128 | 55.1661534153708 | 4.20E-11 | 2.431027E-04 |
| 0.7 | 54.4503884356475 | 54.4503884356913 | 4.38E-11 | 3.238270 E-04 |
| 0.8 | 53.7580890230572 | 53.7580890231646 | 1.07E-10 | 4.139307 E-04 |
| 0.9 | 53.0884858848458 | 53.0884858849116 | 6.58E-11 | 5.127120E-04 |
| 1.0 | 52.4408349486343 | 52.4408349488036 | 1.69E-10 | 6.195049E-04 |

**Problem 4:**

Consider a stiff second order problem

$$y'' = -\lambda^2 y,\ \lambda = 2,\ y(0) = 1,\ y'(0) = 2\ h = 0.01$$

Exact solution: $y(x) = \cos 2x + \sin 2x$

**Table 4. The result of test problem 4**

| X | Exact Solution | Computed Solution | Error in our New Method k=3, h=0.01, P=5 | Error in Adesanya (2011) P = 6, k=5, h=0.01, | Error in Okunuga (2008) P =4, k=1, h=0.01, |
|---|---|---|---|---|---|
| 0.01 | 1.01979867335991 | 1.019798673394 | 3.409E-11 | 2.657762E-11 | - |
| 0.02 | 1.03918944084761 | 1.03918944088 | 3.239E-11 | 1.853988E-10 | 2.65 E – 06 |
| 0.03 | 1.05816454641465 | 1.05816454638 | 3.465E-11 | 4.736862E-10 | 3.98 E – 06 |
| 0.04 | 1.07671640027179 | 1.07671640027203 | 2.40E-13 | 8.885585E-10 | 5.30 E – 06 |
| 0.05 | 1.09483758192485 | 1.09483758192307 | 1.780E-12 | 1.427033E-10 | 6.62 E – 06 |
| 0.06 | 1.11252084314278 | 1.11252084306812 | 7.467E-11 | 2.086028E-09 | 7.94 E – 06 |
| 0.07 | 1.12975911085687 | 1.12975911081783 | 3.904E-11 | 2.862367E-09 | 9.25 E – 06 |
| 0.08 | 1.14654548998987 | 1.14654548994855 | 4.132E-11 | 3.752772E-09 | 1.06 E – 05 |
| 0.09 | 1.16287326621394 | 1.16287326609424 | 1.197E-10 | 4.753892E-09 | 1.19 E – 05 |
| 1.00 | 1.17873590863630 | 1.17873590855288 | 8.342E-11 | 5.862260E-09 | 1.32 E – 05 |

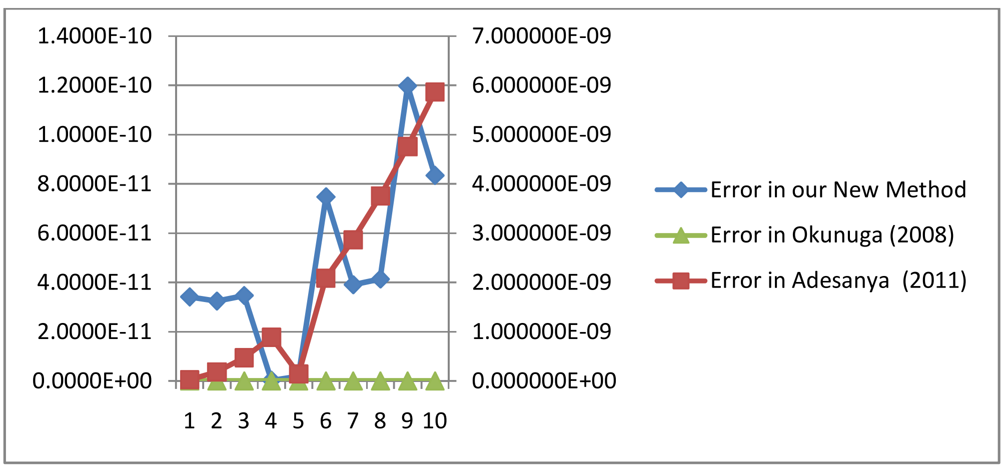


**Fig. 1. Comparison of Error of Okunaga [9], Adesanya [16] and our proposed Method**

## 6. DISCUSSION OF RESULT

We have proposed a direct method for the solution of general second order initial value problems of ordinary differential equations using a three-step single hybrid block method. The results of our new continuous implicit hybrid block method (CIHBM) with step length three and order of accuracy five were compared with other researchers.

In Table 1(a) and 1(b), our method is more accurate than that of Ali Shokri [13] which was executed by of break method.

In Table 2, our new method performs better than Anake [20] which was implemented in block method.

Table 3 shows the comparison of the results of our method with that of Olanegan [21] which was implemented in Taylor series. It is observed that our method is more efficient and accurate than that of Olanegan [21].

Table 4 shows the comparison of the results of our method with that of Okunuga [9] and Adesanya [16]. Okunuga [9] was implemented in Predictor-corrector method while Adesanya [16] was implemented in block method. It is obvious

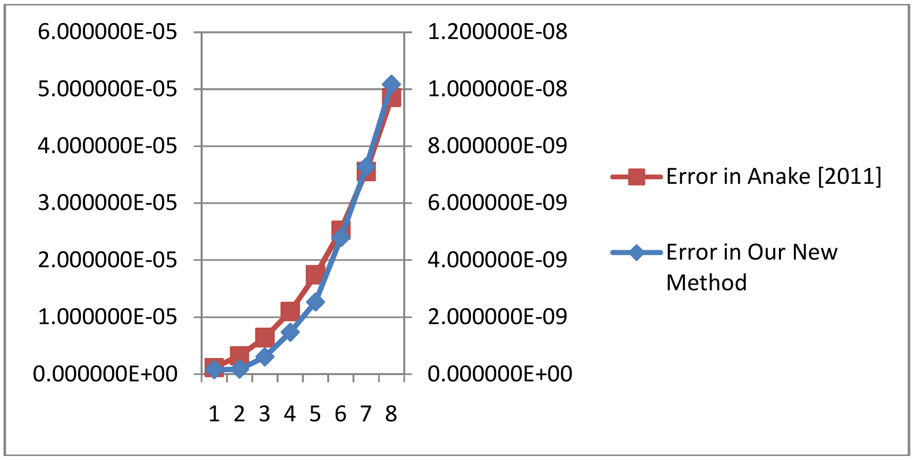


**Fig. 2. Comparison of Error of Anake [20] and our proposed Method**

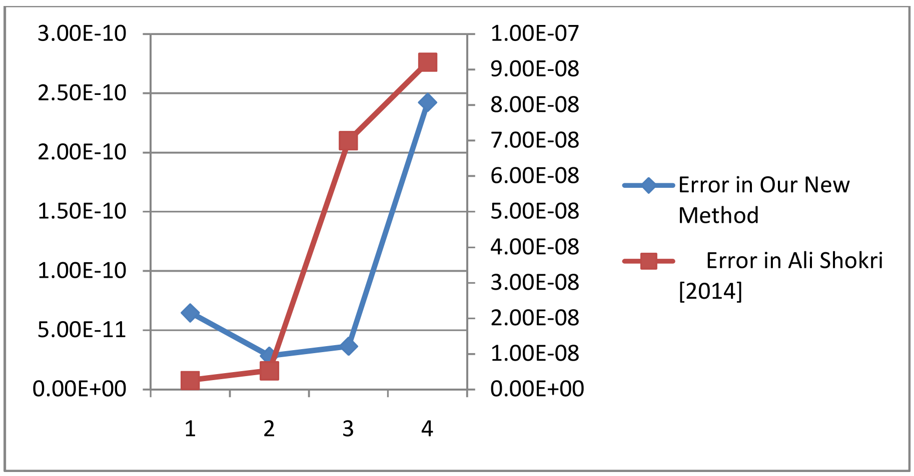


**Fig. 3. Comparison of Error of Ali Shokri [13] and our proposed Method**

that our method is better in accuracy and also efficient than Okunuga [9] and Adesanya [16].

Fig. 1 shows the comparison of Numerical error with Okunuga 2008, adesanya 2011 and our proposed method for problem 4, Fig. 2 shows the comparison of numerical error of Anake 2011 with our method for problem 2 while Fig. 3 shows the comparison of Ali Shokri [13] with our developed method for problem 1. This shows that our developed method compare favourably with the existing method in the literature.

## 7. CONCLUSION

In this article, we have proposed a new continuous implicit hybrid method for numerical treatment of general second order ordinary differential equations initial value problems which was implemented in continuous block method. The results show that our method is more efficient and give better approximation than the existing methods. Hence our method is recommended for the solution of general second order ordinary differential equations initial value problems.

## COMPETING INTERESTS

Authors have declared that no competing interests exist.

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