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A Study on Sum Formulas of Generalized Tetranacci Numbers: Closed Forms of the Sum Formulas $\sum_{k=0}^{n} kW_k$ and $\sum_{k=1}^{n} kW_{-k}$

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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ABSTRACT

In this paper, closed forms of the sum formulas $\sum_{k=0}^{n} kW_k$ and $\sum_{k=1}^{n} kW_{-k}$ for generalized Tetranacci numbers are presented. As special cases, we give summation formulas of Tetranacci, Tetranacci-Lucas, and other fourth-order recurrence sequences.

Keywords: Tetranacci numbers; Tetranacci-Lucas numbers; fourth order Pell numbers; sum formulas; summing formulas.

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1 INTRODUCTION

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. The sequence of Fibonacci numbers $\{F_n\}$ is defined by

$$F_n = F_{n-1} + F_{n-2}, \ \ n \geq 2, \quad F_0 = 0, \ F_1 = 1.$$

The generalization of Fibonacci sequence leads to several nice and interesting sequences. Two

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of these type of sequences are the sequences of Tetranacci and Tetranacci-Lucas which are special case of generalized Tetranacci numbers. A generalized Tetranacci sequence

$$\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3; r, s, t, u)\}_{n\geq 0}$$

is defined by the fourth-order recurrence relations

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}, \tag{1.1}$$

with the initial values W_0, W_1, W_2, W_3 are arbitrary complex (or real) numbers not all being zero and r, s, t, u are real numbers. This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [1,2,3,4,5,6,7].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{t}{u}W_{-(n-1)} - \frac{s}{u}W_{-(n-2)} - \frac{r}{u}W_{-(n-3)} + \frac{1}{u}W_{-(n-4)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

For some specific values of W_0, W_1, W_2, W_3 and r, s, t, u, it is worth presenting these special Tetranacci numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of r, s, t, u and initial values.

In literature, for example, the following names and notations (see Table 1) are used for the special case of r, s, t, u and initial values.

Table 1. A few special case of generalized Tetranacci sequences

No	Sequences (Numbers)	Notation	OEIS [8]	Ref.
1	Tetranacci	$\{M_n\} = \{W_n(0,1,1,2;1,1,1,1)\}$	A000078	[9]
2	Tetranacci-Lucas	$\{R_n\} = \{W_n(4,1,3,7;1,1,1,1)\}$	A073817	[9]
3	fourth order Pell	$\{P_n^{(4)}\} = \{W_n(0,1,2,5;2,1,1,1)\}$	A103142	[10]
4	fourth order Pell-Lucas	$\{Q_n^{(4)}\} = \{W_n(4,2,6,17;2,1,1,1)\}$	A331413	[10]
5	modified fourth order Pell	$\{E_n^{(4)}\} = \{W_n(0,1,1,3;2,1,1,1)\}$	A190139	[10]
6	fourth order Jacobsthal	$\{J_n^{(4)}\} = \{W_n(0,1,1,1;1,1,1,2)\}$	A007909	[11]
7	fourth order Jacobsthal-Lucas	$\{j_n^{(4)}\} = \{W_n(2,1,5,10;1,1,1,2)\}$	A226309	[11]
8	modified fourth order Jacobsthal	$\{K_n^{(4)}\} = \{W_n(3,1,3,10;1,1,1,2)\}$		[11]
9	fourth-order Jacobsthal Perrin	$\{Q_n^{(4)}\} = \{W_n(3,0,2,8;1,1,1,2)\}$		[11]
10	adjusted fourth-order Jacobsthal	$\{S_n^{(4)}\} = \{W_n(0,1,1,2;1,1,1,2)\}$		[11]
11	modified fourth-order Jacobsthal-Lucas	$\{R_n^{(4)}\} = \{W_n(4,1,3,7;1,1,1,2)\}$		[11]
12	4-primes	$\{G_n\} = \{W_n(0,0,1,2;2,3,5,7)\}$		[12]
13	Lucas 4-primes	$\{H_n\} = \{W_n(4,2,10,41;2,3,5,7)\}$		[12]
14	modified 4-primes	$\{E_n\} = \{W_n(0,0,1,1;2,3,5,7)\}$		[12]

Here OEIS stands for On-line Encyclopedia of Integer Sequences. For easy writing, from now on, we drop the superscripts from the sequences, for example we write J_n for $J_n^{(4)}$.

We present some works on summing formulas of the numbers in the following Table 2.

Table 2. A few special study of sum formulas

Name of sequence	Papers which deal with summing formulas
Pell and Pell-Lucas	[13,14,15],[16,17]
Generalized Fibonacci	[18,19,20,21,22,23,24]
Generalized Tribonacci	[25,26,27]
Generalized Tetranacci	[28,29,6]
Generalized Pentanacci	[30,31]
Generalized Hexanacci	[32,33]

The following Theorem present some lineer summing formulas of generalized Tetranacci numbers with positive subscripts.

Theorem 1.1. *For $n \geq 0$ we have the following formulas:*

(a) *If $r+s+t+u-1 \neq 0$, then*

$$\sum_{k=0}^{n} W_k = \frac{\Theta_1}{r+s+t+u-1}$$

(b) *If $(r-s+t-u+1)(r+s+t+u-1) \neq 0$ then*

$$\sum_{k=0}^{n} W_{2k} = \frac{\Theta_2}{(r-s+t-u+1)(r+s+t+u-1)}$$

(c) *If $(r-s+t-u+1)(r+s+t+u-1) \neq 0$ then*

$$\sum_{k=0}^{n} W_{2k+1} = \frac{\Theta_3}{(r-s+t-u+1)(r+s+t+u-1)}$$

where

$\Theta_1 = W_{n+4} + (1-r)W_{n+3} + (1-r-s)W_{n+2} + (1-r-s-t)W_{n+1} - W_3 + (r-1)W_2 + (r+s-1)W_1 + (r+s+t-1)W_0$

$\Theta_2 = (1-s-u)W_{2n+2} + (t+rs+ru)W_{2n+1} + (t^2-u^2+rt-su+u)W_{2n} + (ru+tu)W_{2n-1} - (r+t)W_3 + (s+u+rt+r^2-1)W_2 + (st-ru-t)W_1 + (r^2-s^2+t^2+2s+u+2rt-su-1)W_0$

$\Theta_3 = (r+t)W_{2n+2} + (-s^2+t^2-u^2+rt-2su+s+u)W_{2n+1} + (t+ru-st)W_{2n} - u(s+u-1)W_{2n-1} + (s+u-1)W_3 - (t+rs+ru)W_2 + (r^2-s^2+rt-su+2s+u-1)W_1 - u(r+t)W_0$

Proof. It is given in Soykan [28].

The following Theorem present some linear summing formulas of generalized Tetranacci numbers with negative subscripts.

Theorem 1.2. *For $n \geq 1$ we have the following formulas:*

(a) *If $r+s+t+u-1 \neq 0$, then*

$$\sum_{k=1}^{n} W_{-k} = \frac{\Theta_4}{r+s+t+u-1}$$

(b) *If $(r-s+t-u+1)(r+s+t+u-1) \neq 0$ then*

$$\sum_{k=1}^{n} W_{-2k} = \frac{\Theta_5}{(r-s+t-u+1)(r+s+t+u-1)}$$

(c) *If $(r-s+t-u+1)(r+s+t+u-1) \neq 0$ then*

$$\sum_{k=1}^{n} W_{-2k+1} = \frac{\Theta_6}{(r-s+t-u+1)(r+s+t+u-1)}$$

where

$\Theta_4 = -W_{-n+3} + (r-1)W_{-n+2} + (r+s-1)W_{-n+1} + (r+s+t-1)W_{-n} + W_3 + (1-r)W_2 + (1-r-s)W_1 + (1-s-r-t)W_0$

$\Theta_5 = (s+u-1)W_{-2n+2} - (t+rs+ru)\,W_{-2n+1} + (r^2-s^2+rt-su+2s+u-1)W_{-2n} - u\,(r+t)\,W_{-2n-1} + (r+t)W_3 + (1-r^2-rt-s-u)W_2 + (t+ru-st)\,W_1 + (1-r^2+s^2-t^2-2rt+su-2s-u)W_0$

$\Theta_6 = -(r+t)W_{-2n+2} + (r^2+rt+s+u-1)W_{-2n+1} + (st-t-ru)W_{-2n} + (u^2+su-u)W_{-2n-1} + (1-s-u)W_3 + (t+ru+rs)W_2 + (1-r^2+s^2-rt+su-2s-u)W_1 + u(r+t)W_0$

Proof. It is given in Soykan [28].

In this work, we investigate linear summation formulas of generalized Tetranacci numbers.

2 LINEAR SUM FORMULAS OF GENERALIZED TETRANACCI NUMBERS WITH POSITIVE SUBSCRIPTS

The following Theorem present some lineer summing formulas of generalized Tetranacci numbers with positive subscripts.

Theorem 2.1. *For $n \geq 0$ we have the following formulas:*

(a) *If $r+s+t+u-1 \neq 0$, then*

$$\sum_{k=0}^{n} kW_k = \frac{\Psi_1}{(r+s+t+u-1)^2}$$

(b) *If $(r+s+t+u-1)\,(r-s+t-u+1) \neq 0$ then*

$$\sum_{k=0}^{n} kW_{2k} = \frac{\Psi_2}{(r+s+t+u-1)^2\,(r-s+t-u+1)^2}$$

(c) *If $(r+s+t+u-1)\,(r-s+t-u+1) \neq 0$ then*

$$\sum_{k=0}^{n} kW_{2k+1} = \frac{\Psi_3}{(r+s+t+u-1)^2\,(r-s+t-u+1)^2}$$

where

$\Psi_1 = (-n+2r+s-u+nr+ns+nt+nu-3)W_{n+3} + (4r-n-t-2u-nr^2+2nr+ns+nt+nu-rs+ru-2r^2-nrs-nrt-nru-2)W_{n+2} + (2r-n+2s-2t-3u-nr^2-ns^2+2nr+2ns+nt+nu-2rs+rt+2ru+su-r^2-s^2-2nrs-nrt-nru-nst-nsu-1)W_{n+1} + u(-n+3r+2s+t+nr+ns+nt+nu-4)W_n + (u-s-2r+3)W_3 + (-4r+t+2u+rs-ru+2r^2+2)W_2 + (-2r-2s+2t+3u+2rs-rt-2ru-su+r^2+s^2+1)W_1 + u(4-2s-t-3r)W_0$

$\Psi_2 = (2s-n+u+nr^2-3ns^2+ns^3+nt^2-3nu^2+nu^3-r^2s+st^2-2r^2u-2su^2-s^2u+3ns+3nu-2rt-s^2-2t^2+u^2-u^3-nr^2s-nst^2-nr^2u+3nsu^2+3ns^2u-nt^2u+2nrt-6nsu-2rtu-2nrst-2nrtu-1)W_{2n+2} + (-2t+nt^3+2rs^2+r^3s+r^2t+2ru^2+ru^3+2r^3u+2tu^2-nt-2rs-3ru+2st-t^3+2nrs^2-nrs^3+nr^3s+2nrt^2+nr^2t+2nru^2-ns^2t-nru^3+nr^3u-ntu^2-rst^2+2rsu^2+rs^2u+2r^2tu-nrs-nru+2nst+2ntu+4rsu+2stu+nrst^2+2nr^2st-3nrsu^2-3nrs^2u+nrt^2u+2nr^2tu+4nrsu-2nstu)W_{2n+1} + (-2u-nt^2+nt^4+3nu^2-3nu^3+nu^4+rt^3+r^3t+4st^2+r^2u-6su^2-4s^2u+su^3+s^3u+t^2u+2r^2t^2-3r^2u^2-s^2t^2+2s^2u^2-nu-2rt+5su-3t^2+4u^2-2u^3+3nrt^3+nr^3t+2nst^2+nr^2u-6nsu^2-3ns^2u+3nsu^3+ns^3u+3nt^2u-2r^2su-2rtu^2+3nr^2t^2-nr^2u^2-ns^2t^2+3ns^2u^2-2nt^2u^2-nrt+3nsu+2rst-nrs^2t-nr^2su-3nrtu^2-3nst^2u+2nrst+4nrtu-4nrstu)W_{2n} + u(-2r-3t+nr^3+nt^3+rt^2+2r^2t+2ru^2-s^2t+tu^2-nr-nt+2rs+4st+2tu+r^3-nrs^2+3nrt^2+3nr^2t-nru^2-ns^2t-ntu^2+2nrs+2nru+2nst+2ntu+2rsu-2nrsu-2nstu)W_{2n-1} + (-r^3-2r^2t-2rsu-2rs-rt^2-2ru^2+2r+$

$s^2t-4st-tu^2-2tu+3t)W_3+(-2s-u+3r^2s+2r^3t-st^2+2r^2u+2su^2+s^2u+r^2t^2+2r^2u^2-rt-2r^2+r^4+s^2+2t^2-u^2+u^3-rs^2t+2r^2su+rtu^2+4rst+4rtu+1)W_2+(2t-r^2t-2ru^2+4s^2t-ru^3-2r^3u-s^3t-2tu^2+3ru-5st+t^3+2rst^2+2r^2st+rs^2u-2r^2tu+stu^2-4rsu)W_1+u(-5s-4u+2r^2s+3r^2u-su^2-2s^2u+t^2u+6su-r^2+4s^2-s^3+t^2+2u^2+2rst+4rtu+2)W_0$

$\Psi_3=(-r-2t+nr^3+nt^3+rs^2-2rt^2-r^2t+3ru^2+2tu^2-nr-nt-2ru+2st-t^3-nrs^2+3nrt^2+3nr^2t-nru^2-ns^2t-ntu^2+2nrs+2nru+2nst+2ntu+4rsu+2stu-2nrsu-2nstu)W_{2n+2}+(3ns^2-2u-s-3ns^3+ns^4-nt^2+nt^4+3nu^2-3nu^3+nu^4+rt^3+r^3t+2st^2+r^2u-5su^2-4s^2u+t^2u-r^2s^2+2r^2t^2-3r^2u^2-ns-nu-2rt+6su+2s^2-s^3-3t^2+4u^2-2u^3+nr^2s+3nrt^3+nr^3t+3nst^2+nr^2u-9nsu^2-9ns^2u+4nsu^3+4ns^3u+3nt^2u-4r^2su-2rtu^2-nr^2s^2+3nr^2t^2-nr^2u^2-2ns^2t^2+6ns^2u^2-2nt^2u^2-nrt+6nsu-3nrs^2t-2nr^2su-3nrtu^2-4nst^2u+4nrst+4nrtu-2rstu-6nrstu)W_{2n+1}+(-t+nt^3-2rt^2-s^2t+2ru^3+st^3+r^3u+3tu^2-nt-2ru+2st-2tu-2t^3+2nrt^2+nr^2t+2nru^2-3ns^2t-nru^3-nst^3+nr^3u+ns^3t-ntu^2-r^2st+2rsu^2-rt^2u-2stu^2-2s^2tu-nru+3nst+2ntu+2rsu+4stu-2nrst^2-nr^2st-2nrsu^2-nrs^2u+nrt^2u+2nr^2tu+nstu^2+2ns^2tu+2nrsu-4nstu)W_{2n}+u(2s-n+u+nr^2-3ns^2+ns^3+nt^2-3nu^2+nu^3-r^2s+st^2-2r^2u-2su^2-s^2u+3ns+3nu-2rt-s^2-2t^2+u^2-u^3-nr^2s-nst^2-nr^2u+3nsu^2+3ns^2u-nt^2u+2nrt-6nsu-2rtu-2nrst-2nrtu-1)W_{2n-1}+(-2s-u+r^2s-st^2+2r^2u+2su^2+s^2u+2rt+s^2+2t^2-u^2+u^3+2rtu+1)W_3+(2t-2rs^2-r^3s-r^2t-2ru^2-ru^3-2r^3u-2tu^2+2rs+3ru-2st+t^3+rst^2-2rsu^2-rs^2u-2r^2tu-4rsu-2stu)W_2+(2u-rt^3-r^3t-4st^2-r^2u+6su^2+4s^2u-su^3-s^3u-t^2u-2r^2t^2+3r^2u^2+s^2t^2-2s^2u^2+2rt-5su+3t^2-4u^2+2u^3+2r^2su+2rtu^2-2rst)W_1+u(-r^3-2r^2t-2rsu-2rs-rt^2-2ru^2+2r+s^2t-4st-tu^2-2tu+3t)W_0$

Proof.

(a) Using the recurrence relation

$$W_n=rW_{n-1}+sW_{n-2}+tW_{n-3}+uW_{n-4}$$

i.e.

$$uW_{n-4}=W_n-rW_{n-1}-sW_{n-2}-tW_{n-3}$$

we obtain

$$\begin{aligned}
u\times 0\times W_0 &= 0\times W_4-r\times 0\times W_3-s\times 0\times W_2-t\times 0\times W_1\\
u\times 1\times W_1 &= 1\times W_5-r\times 1\times W_4-s\times 1\times W_3-t\times 1\times W_2\\
u\times 2\times W_2 &= 2\times W_6-r\times 2\times W_5-s\times 2\times W_4-t\times 2\times W_3\\
u\times 3\times W_3 &= 3\times W_7-r\times 3\times W_6-s\times 3\times W_5-t\times 3\times W_4\\
&\vdots\\
u(n-4)W_{n-4} &= (n-4)W_n-r(n-4)W_{n-1}-s(n-4)W_{n-2}-t(n-4)W_{n-3}\\
u(n-3)W_{n-3} &= (n-3)W_{n+1}-r(n-3)W_n-s(n-3)W_{n-1}-t(n-3)W_{n-2}\\
u(n-2)W_{n-2} &= (n-2)W_{n+2}-r(n-2)W_{n+1}-s(n-2)W_n-t(n-2)W_{n-1}\\
u(n-1)W_{n-1} &= (n-1)W_{n+3}-r(n-1)W_{n+2}-s(n-1)W_{n+1}-t(n-1)W_n\\
u\times n\times W_n &= n\times W_{n+4}-r\times n\times W_{n+3}-s\times n\times W_{n+2}-t\times n\times W_{n+1}
\end{aligned}$$

If we add the equations side by side (and using Theorem 1.1 (a)), we get (a)

(b) and (c) Using the recurrence relation

$$W_n=rW_{n-1}+sW_{n-2}+tW_{n-3}+uW_{n-4}$$

i.e.

$$rW_{n-1}=W_n-sW_{n-2}-tW_{n-3}-uW_{n-4}$$

we obtain

$$\begin{aligned}
r \times 1 \times W_3 &= 1 \times W_4 - s \times 1 \times W_2 - t \times 1 \times W_1 - u \times 1 \times W_0 \\
r \times 2 \times W_5 &= 2 \times W_6 - s \times 2 \times W_4 - t \times 2 \times W_3 - u \times 2 \times W_2 \\
r \times 3 \times W_7 &= 3 \times W_8 - s \times 3 \times W_6 - t \times 3 \times W_5 - u \times 3 \times W_4 \\
r \times 4 \times W_9 &= 4 \times W_{10} - s \times 4 \times W_8 - t \times 4 \times W_7 - u \times 4 \times W_6 \\
&\vdots \\
r(n-1)W_{2n-1} &= (n-1)W_{2n} - s(n-1)W_{2n-2} - t(n-1)W_{2n-3} - u(n-1)W_{2n-4} \\
rnW_{2n+1} &= nW_{2n+2} - snW_{2n} - tnW_{2n-1} - unW_{2n-2} \\
r(n+1)W_{2n+3} &= (n+1)W_{2n+4} - s(n+1)W_{2n+2} - t(n+1)W_{2n+1} - u(n+1)W_{2n}
\end{aligned}$$

Now, if we add the above equations side by side, we get

$$\begin{aligned}
r(-0 \times W_1 + \sum_{k=0}^{n} kW_{2k+1}) &= (nW_{2n+2} - 0 \times W_2 - (-1)W_0 + \sum_{k=0}^{n} (k-1)W_{2k}) \\
&\quad -s(-0 \times W_0 + \sum_{k=0}^{n} kW_{2k}) - t(-(n+1)W_{2n+1} + \sum_{k=0}^{n} (k+1)W_{2k+1}) \\
&\quad -u(-(n+1)W_{2n} + \sum_{k=0}^{n} (k+1)W_{2k})
\end{aligned} \tag{2.1}$$

Similarly, using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3} - uW_{n-4}$$

we write the following obvious equations;

$$\begin{aligned}
r \times 1 \times W_2 &= 1 \times W_3 - s \times 1 \times W_1 - t \times 1 \times W_0 - u \times 1 \times W_{-1} \\
r \times 2 \times W_4 &= 2 \times W_5 - s \times 2 \times W_3 - t \times 2 \times W_2 - u \times 2 \times W_1 \\
r \times 3 \times W_6 &= 3 \times W_7 - s \times 3 \times W_5 - t \times 3 \times W_4 - u \times 3 \times W_3 \\
r \times 8 \times W_8 &= 4 \times W_9 - s \times 8 \times W_7 - t \times 8 \times W_6 - u \times 8 \times W_5 \\
&\vdots \\
r(n-1)W_{2n-2} &= (n-1)W_{2n-1} - s(n-1)W_{2n-3} - t(n-1)W_{2n-4} - u(n-1)W_{2n-5} \\
rnW_{2n} &= nW_{2n+1} - snW_{2n-1} - tnW_{2n-2} - unW_{2n-3} \\
r(n+1)W_{2n+2} &= (n+1)W_{2n+3} - s(n+1)W_{2n+1} - t(n+1)W_{2n} - u(n+1)W_{2n-1}
\end{aligned}$$

Now, if we add the above equations side by side, we obtain

$$\begin{aligned}
r(-0 \times W_0 + \sum_{k=0}^{n} kW_{2k}) &= (-0 \times W_1 + \sum_{k=0}^{n} kW_{2k+1}) - s(-(n+1)W_{2n+1} + \sum_{k=0}^{n} (k+1)W_{2k+1}) \\
&\quad -t(-(n+1)W_{2n} + \sum_{k=0}^{n} (k+1)W_{2k}) - u(-(n+2)W_{2n+1} \\
&\quad -(n+1)W_{2n-1} + 1 \times W_{-1} + \sum_{k=0}^{n} (k+2)W_{2k+1})
\end{aligned}$$

Since

$$W_{-1}=-\frac{t}{u}W_0-\frac{s}{u}W_1-\frac{r}{u}W_2+\frac{1}{u}W_3$$

we have

$$\begin{aligned} r(\sum_{k=0}^{n}kW_{2k}) &= (\sum_{k=0}^{n}kW_{2k+1})-s(-(n+1)W_{2n+1}+\sum_{k=0}^{n}kW_{2k+1}+\sum_{k=0}^{n}W_{2k+1}) \\ &\quad -t(-(n+1)W_{2n}+\sum_{k=0}^{n}kW_{2k}+\sum_{k=0}^{n}W_{2k}) \\ &\quad -u(-(n+2)W_{2n+1}-(n+1)W_{2n-1} \\ &\quad +(-\frac{t}{u}W_0-\frac{s}{u}W_1-\frac{r}{u}W_2+\frac{1}{u}W_3)+\sum_{k=0}^{n}kW_{2k+1}+2\sum_{k=0}^{n}W_{2k+1}). \end{aligned} \tag{2.2}$$

Then, solving the system (2.1)-(2.2) (using Theorem 1.1 (b) and (c)), the required result of (b) and (c) follow.

Taking $r=s=t=u=1$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the following proposition.

Proposition 2.1. *If $r=s=t=u=1$ then for $n\geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}kW_k=\frac{1}{9}((3n-1)W_{n+3}-3W_{n+2}-(3n+2)W_{n+1}+(3n+2)W_n+W_3+3W_2+2W_1-2W_0)$.

(b) $\sum_{k=0}^{n}kW_{2k}=\frac{1}{9}(-(3n+10)W_{2n+2}+(9n+15)W_{2n+1}+(3n-2)W_{2n}+(6n+11)W_{2n-1}-11W_3+21W_2-4W_1+13W_0)$.

(c) $\sum_{k=0}^{n}kW_{2k+1}=\frac{1}{9}((6n+5)W_{2n+2}-12W_{2n+1}+(3n+1)W_{2n}-(3n+10)W_{2n-1}+10W_3-15W_2-11W_0+2W_1)$.

From the above proposition, we have the following corollary which gives linear sum formulas of Tetranacci numbers (take $W_n=M_n$ with $M_0=0,M_1=1,M_2=1,M_3=2$).

Corollary 2.2. *For $n\geq 0$, Tetranacci numbers have the following properties.*

(a) $\sum_{k=0}^{n}kM_k=\frac{1}{9}((3n-1)M_{n+3}-3M_{n+2}-(3n+2)M_{n+1}+(3n+2)M_n+7)$.

(b) $\sum_{k=0}^{n}kM_{2k}=\frac{1}{9}(-(3n+10)M_{2n+2}+(9n+15)M_{2n+1}+(3n-2)M_{2n}+(6n+11)M_{2n-1}-5)$.

(c) $\sum_{k=0}^{n}kM_{2k+1}=\frac{1}{9}((6n+5)M_{2n+2}-12M_{2n+1}+(3n+1)M_{2n}-(3n+10)M_{2n-1}+7)$.

Taking $W_n=R_n$ with $R_0=4,R_1=1,R_2=3,R_3=7$ in the above proposition, we have the following corollary which presents linear sum formulas of Tetranacci-Lucas numbers.

Corollary 2.3. *For $n\geq 0$, Tetranacci-Lucas numbers have the following properties.*

(a) $\sum_{k=0}^{n}kR_k=\frac{1}{9}((3n-1)R_{n+3}-3R_{n+2}-(3n+2)R_{n+1}+(3n+2)R_n+10)$.

(b) $\sum_{k=0}^{n}kR_{2k}=\frac{1}{9}(-(3n+10)R_{2n+2}+(9n+15)R_{2n+1}+(3n-2)R_{2n}+(6n+11)R_{2n-1}+34)$.

(c) $\sum_{k=0}^{n}kR_{2k+1}=\frac{1}{9}((6n+5)R_{2n+2}-12R_{2n+1}+(3n+1)R_{2n}-(3n+10)R_{2n-1}-17)$.

Taking $r=2,s=t=u=1$ in Theorem 2.1 (a), (b) and (c), we obtain the following proposition.

Proposition 2.2. *If $r=2,s=t=u=1$ then for $n\geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}kW_k=\frac{1}{16}((4n+1)W_{n+3}-(4n+5)W_{n+2}-(8n+2)W_{n+1}+(4n+5)W_n-W_3+5W_2+2W_1-5W_0)$.

(b) $\sum_{k=0}^{n}kW_{2k}=\frac{1}{64}(-(8n+23)W_{2n+2}+(40n+51)W_{2n+1}+(16n-2)W_{2n}+(24n+29)W_{2n-1}-29W_3+81W_2-22W_1+31W_0)$.

(c) $\sum_{k=0}^{n} kW_{2k+1} = \frac{1}{64}((24n+5)W_{2n+2} + (8n-25)W_{2n+1} + (16n+6)W_{2n} - (8n+23)W_{2n-1} + 23W_3 - 51W_2 + 2W_1 - 29W_0)$.

From the last proposition, we have the following corollary which gives linear sum formulas of fourth-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5$).

Corollary 2.4. *For $n \geq 0$, fourth-order Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n} kP_k = \frac{1}{16}((4n+1)P_{n+3} - (4n+5)P_{n+2} - (8n+2)P_{n+1} + (4n+5)P_n + 7)$.

(b) $\sum_{k=0}^{n} kP_{2k} = \frac{1}{64}(-(8n+23)P_{2n+2} + (40n+51)P_{2n+1} + (16n-2)P_{2n} + (24n+29)P_{2n-1} - 5)$.

(c) $\sum_{k=0}^{n} kP_{2k+1} = \frac{1}{64}((24n+5)P_{2n+2} + (8n-25)P_{2n+1} + (16n+6)P_{2n} - (8n+23)P_{2n-1} + 15)$.

Taking $W_n = Q_n$ with $Q_0 = 4, Q_1 = 2, Q_2 = 6, Q_3 = 17$ in the last proposition, we have the following corollary which presents linear sum formulas of fourth-order Pell-Lucas numbers.

Corollary 2.5. *For $n \geq 0$, fourth-order Pell-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} kQ_k = \frac{1}{16}((4n+1)Q_{n+3} - (4n+5)Q_{n+2} - (8n+2)Q_{n+1} + (4n+5)Q_n - 3)$.

(b) $\sum_{k=0}^{n} kQ_{2k} = \frac{1}{64}(-(8n+23)Q_{2n+2} + (40n+51)Q_{2n+1} + (16n-2)Q_{2n} + (24n+29)Q_{2n-1} + 73)$.

(c) $\sum_{k=0}^{n} kQ_{2k+1} = \frac{1}{64}((24n+5)Q_{2n+2} + (8n-25)Q_{2n+1} + (16n+6)Q_{2n} - (8n+23)Q_{2n-1} - 27)$.

If $r = 1, s = 1, t = 1, u = 2$ then $(r+s+t+u-1)(r-s+t-u+1) = 0$ so we can't use Theorem 2.1 (b) and (c), directly.

Proposition 2.3. *If $r = 1, s = 1, t = 1, u = 2$ then for $n \geq 0$ we have the following formula:*

$$\sum_{k=0}^{n} kW_k = \frac{1}{16}((4n-2)W_{n+3} - 4W_{n+2} - (4n+2)W_{n+1} + 2(4n+2)W_n + 2W_3 + 4W_2 + 2W_1 - 4W_0).$$

Taking $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1$ in the last proposition, we have the following corollary which presents linear sum formula of fourth-order Jacobsthal numbers.

Corollary 2.6. *For $n \geq 0$, fourth order Jacobsthal numbers have the following property:*

$$\sum_{k=0}^{n} kJ_k = \frac{1}{16}((4n-2)J_{n+3} - 4J_{n+2} - (4n+2)J_{n+1} + 2(4n+2)J_n + 8).$$

From the last proposition, we have the following corollary which gives linear sum formula of fourth order Jacobsthal-Lucas numbers (take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10$).

Corollary 2.7. *For $n \geq 0$, fourth order Jacobsthal-Lucas numbers have the following property:*

$$\sum_{k=0}^{n} kj_k = \frac{1}{16}((4n-2)j_{n+3} - 4j_{n+2} - (4n+2)j_{n+1} + 2(4n+2)j_n + 34).$$

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10$ in the last proposition, we have the following corollary which presents linear sum formula of modified fourth order Jacobsthal numbers.

Corollary 2.8. *For $n \geq 0$,modified fourth order Jacobsthal numbers have the following property:*

$$\sum_{k=0}^{n} kK_k = \frac{1}{16}((4n-2)K_{n+3} - 4K_{n+2} - (4n+2)K_{n+1} + 2(4n+2)K_n + 22).$$

From the last proposition, we have the following corollary which gives linear sum formula of fourth-order Jacobsthal Perrin numbers (take $W_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8$).

Corollary 2.9. *For $n \geq 0$, fourth-order Jacobsthal Perrin numbers have the following property:*

$$\sum_{k=0}^{n} kQ_k = \frac{1}{16}((4n-2)Q_{n+3} - 4Q_{n+2} - (4n+2)Q_{n+1} + 2(4n+2)Q_n + 12).$$

Taking $W_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2$ in the last proposition, we have the following corollary which presents linear sum formula of adjusted fourth-order Jacobsthal numbers.

Corollary 2.10. *For $n \geq 0$, adjusted fourth-order Jacobsthal numbers have the following property:*

$$\sum_{k=0}^{n} kS_k = \frac{1}{16}((4n-2)S_{n+3} - 4S_{n+2} - (4n+2)S_{n+1} + 2(4n+2)S_n + 10).$$

From the last proposition, we have the following corollary which gives linear sum formula of modified fourth-order Jacobsthal-Lucas numbers (take $W_n = R_n$ with $R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7$).

Corollary 2.11. *For $n \geq 0$, modified fourth-order Jacobsthal-Lucas numbers have the following property:*

$$\sum_{k=0}^{n} kR_k = \frac{1}{16}((4n-2)R_{n+3} - 4R_{n+2} - (4n+2)R_{n+1} + 2(4n+2)R_n + 12).$$

Taking $r = 2, s = 3, t = 5, u = 7$ in Theorem 2.1 (a), (b) and (c), we obtain the following proposition.

Proposition 2.4. *If $r = 2, s = 3, t = 5, u = 7$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} kW_k = \frac{1}{256}((16n-3)W_{n+3} - (16n+13)W_{n+2} - (64n-12)W_{n+1} + 7(16n+13)W_n + 3W_3 + 13W_2 - 12W_1 - 91W_0)$.

(b) $\sum_{k=0}^{n} kW_{2k} = \frac{1}{1024}((288n-851)W_{2n+2} - (800n-2627)W_{2n+1} + (896n-436)W_{2n} - 7(224n-701) W_{2n-1} - 701W_3 + 2253W_2 - 524W_1 + 3941W_0)$.

(c) $\sum_{k=0}^{n} kW_{2k+1} = \frac{1}{1024}(-(224n-925)W_{2n+2} + (1760n-2989)W_{2n+1} - (128n-652)W_{2n} + 7(288n-851)W_{2n-1} + 851W_3 - 2627W_2 + 436W_1 - 4907W_0)$.

From the last proposition, we have the following corollary which gives linear sum formulas of 4-primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 1, G_3 = 2$).

Corollary 2.12. *For $n \geq 0$, 4-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n} kG_k = \frac{1}{256}((16n-3)G_{n+3} - (16n+13)G_{n+2} - (64n-12)G_{n+1} + 7(16n+13)G_n + 19)$.

(b) $\sum_{k=0}^{n} kG_{2k} = \frac{1}{1024}((288n-851)G_{2n+2} - (800n-2627)G_{2n+1} + (896n-436)G_{2n} - 7(224n-701) G_{2n-1} + 851)$.

(c) $\sum_{k=0}^{n} kG_{2k+1} = \frac{1}{1024}(-(224n-925)G_{2n+2} + (1760n-2989)G_{2n+1} - (128n-652)G_{2n} + 7(288n-851)G_{2n-1} - 925)$.

Taking $W_n = H_n$ with $H_0 = 4, H_1 = 2, H_2 = 10, H_3 = 41$ in the last proposition, we have the following corollary which presents linear sum formulas of Lucas 4-primes numbers.

Corollary 2.13. *For $n \geq 0$, Lucas 4-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n} kH_k = \frac{1}{256}((16n-3)H_{n+3} - (16n+13)H_{n+2} - (64n-12)H_{n+1} + 7(16n+13)H_n - 135)$.

(b) $\sum_{k=0}^{n} kH_{2k} = \frac{1}{1024}((288n-851)H_{2n+2} - (800n-2627)H_{2n+1} + (896n-436)H_{2n} - 7(224n-701)H_{2n-1} + 8505)$.

(c) $\sum_{k=0}^{n} kH_{2k+1} = \frac{1}{1024}(-(224n-925)H_{2n+2} + (1760n-2989)H_{2n+1} - (128n-652)H_{2n} + 7(288n-851)H_{2n-1} - 10135)$.

From the last proposition, we have the following corollary which gives linear sum formulas of modified 4-primes numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 1, E_3 = 1$).

Corollary 2.14. *For $n \geq 0$, modified 4-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n} kE_k = \frac{1}{256}((16n-3)E_{n+3} - (16n+13)E_{n+2} - (64n-12)E_{n+1} + 7(16n+13)E_n + 16)$.

(b) $\sum_{k=0}^{n} kE_{2k} = \frac{1}{1024}((288n-851)E_{2n+2} - (800n-2627)E_{2n+1} + (896n-436)E_{2n} - 7(224n-701)E_{2n-1} + 1552)$.

(c) $\sum_{k=0}^{n} kE_{2k+1} = \frac{1}{1024}(-(224n-925)E_{2n+2} + (1760n-2989)E_{2n+1} - (128n-652)E_{2n} + 7(288n-851)E_{2n-1} - 1776)$.

3 LINEAR SUM FORMULAS OF GENERALIZED TETRANACCI NUMBERS WITH NEGATIVE SUBSCRIPTS

The following Theorem present some linear summing formulas of generalized Tetranacci numbers with negative subscripts.

Theorem 3.1. *For $n \geq 1$ we have the following formulas:*

(a) *If $r+s+t+u-1 \neq 0$, then*

$$\sum_{k=1}^{n} kW_{-k} = \frac{\Psi_4}{(r+s+t+u-1)^2}$$

(b) *If$(r+s+t+u-1)(r-s+t-u+1) \neq 0$ then*

$$\sum_{k=1}^{n} kW_{-2k} = \frac{\Psi_5}{(r+s+t+u-1)^2 (r-s+t-u+1)^2}$$

(c) *If $(r+s+t+u-1)(r-s+t-u+1) \neq 0$ then*

$$\sum_{k=1}^{n} kW_{-2k+1} = \frac{\Psi_6}{(r+s+t+u-1)^2 (r-s+t-u+1)^2}$$

where

$\Psi_4 = (n+2r+s-u-nr-ns-nt-nu-3)W_{-n+3} + (n+4r-t-2u+nr^2-2nr-ns-nt-nu-rs+ru-2r^2+nrs+nrt+nru-2)W_{-n+2} + (n+2r+2s-2t-3u+nr^2+ns^2-2nr-2ns-nt-nu-2rs+rt+2ru+su-r^2-s^2+2nrs+nrt+nru+nst+nsu-1)W_{-n+1} + (n-4u+nr^2+ns^2+nt^2-2nr-2ns-2nt-nu+3ru+2su+tu+2nrs+2nrt+nru+2nst+nsu+ntu)W_{-n} + (u-s-2r+3)W_3 + (-4r+t+2u+rs-ru+2r^2+2)W_2 + (-2r-2s+2t+3u+2rs-rt-2ru-su+r^2+s^2+1)W_1 + u(4-2s-t-3r)W_0$

$\Psi_5 = (n+2s+u-nr^2+3ns^2-ns^3-nt^2+3nu^2-nu^3-r^2s+st^2-2r^2u-2su^2-s^2u-3ns-3nu-2rt-s^2-2t^2+u^2-u^3+nr^2s+nst^2+nr^2u-3nsu^2-3ns^2u+nt^2u-2nrt+6nsu-2rtu+2nrst+2nrtu-1)W_{-2n+2} + (2rs^2-nt^3-2t+r^3s+r^2t+2ru^2+ru^3+2r^3u+2tu^2+nt-2rs-3ru+2st-t^3-2nrs^2+nrs^3-nr^3s-2nrt^2-nr^2t-2nru^2+ns^2t+nru^3-nr^3u+ntu^2-rst^2+2rsu^2+$

$rs^2u+2r^2tu+nrs+nru-2nst-2ntu+4rsu+2stu-nrst^2-2nr^2st+3nrsu^2+3nrs^2u-nrt^2u-2nr^2tu-4nrsu+2nstu)W_{-2n+1}+(n-2u-2nr^2+nr^4+6ns^2-4ns^3+ns^4-nt^2+3nu^2-nu^3+rt^3+r^3t+4st^2+r^2u-6su^2-4s^2u+su^3+s^3u+t^2u+2r^2t^2-3r^2u^2-s^2t^2+2s^2u^2-4ns-3nu-2rt+5su-3t^2+4u^2-2u^3+4nr^2s+nrt^3+3nr^3t+2nst^2+3nr^2u-6nsu^2-9ns^2u+nsu^3+3ns^3u+nt^2u-2r^2su-2rtu^2-2nr^2s^2+3nr^2t^2-nr^2u^2-ns^2t^2+3ns^2u^2-3nrt+9nsu+2rst-3nrs^2t-3nr^2su-nrtu^2-nst^2u+6nrst+4nrtu-4nrstu)W_{-2n}+u(-2r-3t-nr^3-nt^3+rt^2+2r^2t+2ru^2-s^2t+tu^2+nr+nt+2rs+4st+2tu+r^3+nrs^2-3nrt^2-3nr^2t+nru^2+ns^2t+ntu^2-2nrs-2nru-2nst-2ntu+2rsu+2nrsu+2nstu)W_{-2n-1}+(-r^3-2r^2t-2rsu-2rs-rt^2-2ru^2+2r+s^2t-4st-tu^2-2tu+3t)W_3++(-2s-u+3r^2s+2r^3t-st^2+2r^2u+2su^2+s^2u+r^2t^2+2r^2u^2-rt-2r^2+r^4+s^2+2t^2-u^2+u^3-rs^2t+2r^2su+rtu^2+4rst+4rtu+1)W_2+(2t-r^2t-2ru^2+4s^2t-ru^3-2r^3u-s^3t-2tu^2+3ru-5st+t^3+2rst^2+2r^2st+rs^2u-2r^2tu+stu^2-4rsu)W_1+u(-5s-4u+2r^2s+3r^2u-su^2-2s^2u+t^2u+6su-r^2+4s^2-s^3+t^2+2u^2+2rst+4rtu+2)W_0$

$\Psi_6=(rs^2-2t-nr^3-nt^3-r-2rt^2-r^2t+3ru^2+2tu^2+nr+nt-2ru+2st-t^3+nrs^2-3nrt^2-3nr^2t+nru^2+ns^2t+ntu^2-2nrs-2nru-2nst-2ntu+4rsu+2stu+2nrsu+2nstu)W_{-2n+2}+(n-s-2u-2nr^2+nr^4+3ns^2-ns^3-nt^2+3nu^2-nu^3+rt^3+r^3t+2st^2+r^2u-5su^2-4s^2u+t^2u-r^2s^2+2r^2t^2-3r^2u^2-3ns-3nu-2rt+6su+2s^2-s^3-3t^2+4u^2-2u^3+3nr^2s+nrt^3+3nr^3t+nst^2+3nr^2u-3nsu^2-3ns^2u+nt^2u-4r^2su-2rtu^2-nr^2s^2+3nr^2t^2-nr^2u^2-3nrt+6nsu-nrs^2t-2nr^2su-nrtu^2+4nrst+4nrtu-2rstu-2nrstu)W_{-2n+1}+(2ru^3-nt^3-2rt^2-s^2t-t+st^3+r^3u+3tu^2+nt-2ru+2st-2tu-2t^3-2nrt^2-nr^2t-2nru^2+3ns^2t+nru^3+nst^3-nr^3u-ns^3t+ntu^2-r^2st+2rsu^2-rt^2u-2stu^2-2s^2tu+nru-3nst-2ntu+2rsu+4stu+2nrst^2+nr^2st+2nrsu^2+nrs^2u-nrt^2u-2nr^2tu-nstu^2-2ns^2tu-2nrsu+4nstu)W_{-2n}+u(n+2s+u-nr^2+3ns^2-ns^3-nt^2+3nu^2-nu^3-r^2s+st^2-2r^2u-2su^2-s^2u-3ns-3nu-2rt-s^2-2t^2+u^2-u^3+nr^2s+nst^2+nr^2u-3nsu^2-3ns^2u+nt^2u-2nrt+6nsu-2rtu+2nrst+2nrtu-1)W_{-2n-1}+(-2s-u+r^2s-st^2+2r^2u+2su^2+s^2u+2rt+s^2+2t^2-u^2+u^3+2rtu+1)W_3+(2t-2rs^2-r^3s-r^2t-2ru^2-ru^3-2r^3u-2tu^2+2rs+3ru-2st+t^3+rst^2-2rsu^2-rs^2u-2r^2tu-4rsu-2stu)W_2+(2u-rt^3-r^3t-4st^2-r^2u+6su^2+4s^2u-su^3-s^3u-t^2u-2r^2t^2+3r^2u^2+s^2t^2-2s^2u^2+2rt-5su+3t^2-4u^2+2u^3+2r^2su+2rtu^2-2rst)W_1+u(-r^3-2r^2t-2rsu-2rs-rt^2-2ru^2+2r+s^2t-4st-tu^2-2tu+3t)W_0$

Proof.

(a) Using the recurrence relation

$$W_{-n+4}=rW_{-n+3}+sW_{-n+2}+tW_{-n+1}+uW_{-n}$$

i.e.

$$uW_{-n}=W_{-n+4}-rW_{-n+3}-sW_{-n+2}-tW_{-n+1}$$

we obtain

$$\begin{array}{rcl}
unW_{-n} & = & nW_{-n+4}-rnW_{-n+3}-snW_{-n+2}-tnW_{-n+1} \\
u(n-1)W_{-n+1} & = & (n-1)W_{-n+5}-r(n-1)W_{-n+4}-s(n-1)W_{-n+3}-t(n-1)W_{-n+2} \\
u(n-2)W_{-n+2} & = & (n-2)W_{-n+6}-r(n-2)W_{-n+5}-s(n-2)W_{-n+4}-t(n-2)W_{-n+3} \\
 & & \vdots \\
u\times 5\times W_{-5} & = & 5\times W_{-1}-r\times 5\times W_{-2}-s\times 5\times W_{-3}-t\times 5\times W_{-4} \\
u\times 4\times W_{-4} & = & 4\times W_{0}-r\times 4\times W_{-1}-s\times 4\times W_{-2}-t\times 4\times W_{-3} \\
u\times 3\times W_{-3} & = & 3\times W_{1}-r\times 3\times W_{0}-s\times 3\times W_{-1}-t\times 3\times W_{-2} \\
u\times 2\times W_{-2} & = & 2\times W_{2}-r\times 2\times W_{1}-s\times 2\times W_{0}-t\times 2\times W_{-1} \\
u\times 1\times W_{-1} & = & 1\times W_{3}-r\times 1\times W_{2}-s\times 1\times W_{1}-t\times 1\times W_{0}.
\end{array}$$

If we add the above equations side by side (and using Theorem 1.2 (a)), we get (a)

(b) and (c) Using the recurrence relation

$$W_{-n+4} = rW_{-n+3} + sW_{-n+2} + tW_{-n+1} + uW_{-n}$$

i.e.

$$tW_{-n+1} = W_{-n+4} - rW_{-n+3} - sW_{-n+2} - uW_{-n}$$

we obtain

$$\begin{aligned}
tnW_{-2n+1} &= nW_{-2n+4} - rnW_{-2n+3} - snW_{-2n+2} - unW_{-2n} \\
t(n-1)W_{-2n+3} &= (n-1)W_{-2n+6} - r(n-1)W_{-2n+5} - s(n-1)W_{-2n+4} - u(n-1)W_{-2n+2} \\
t(n-2)W_{-2n+5} &= (n-2)W_{-2n+8} - r(n-2)W_{-2n+7} - s(n-2)W_{-2n+6} - u(n-2)W_{-2n+4} \\
t(n-3)W_{-2n+7} &= (n-3)W_{-2n+10} - r(n-3)W_{-2n+9} - s(n-3)W_{-2n+8} - u(n-3)W_{-2n+6} \\
&\vdots \\
t \times 3 \times W_{-5} &= 3 \times W_{-2} - r \times 3 \times W_{-3} - s \times 3 \times W_{-4} - u \times 3 \times W_{-6} \\
t \times 2 \times W_{-3} &= 2 \times W_{0} - r \times 2 \times W_{-1} - s \times 2 \times W_{-2} - u \times 2 \times W_{-4} \\
t \times 1 \times W_{-1} &= 1 \times W_{2} - r \times 1 \times W_{1} - s \times 1 \times W_{0} - u \times 1 \times W_{-2}.
\end{aligned}$$

If we add the above equations by side by, we get

$$\begin{aligned}
t\sum_{k=1}^{n} kW_{-2k+1} &= (-(n+1)W_{-2n+2} - (n+2)W_{-2n} + 2 \times W_0 + W_2 + \sum_{k=1}^{n}(k+2)W_{-2k}) \quad (3.1) \\
&\quad -r(-(n+1)W_{-2n+1} + W_1 + \sum_{k=1}^{n}(k+1)W_{-2k+1}) \\
&\quad -s(-(n+1)W_{-2n} + W_0 + \sum_{k=1}^{n}(k+1)W_{-2k}) - u(\sum_{k=1}^{n} kW_{-2k}).
\end{aligned}$$

Similarly, using the recurrence relation

$$W_{-n+4} = rW_{-n+3} + sW_{-n+2} + tW_{-n+1} + uW_{-n}$$

i.e.

$$tW_{-n} = W_{-n+3} - rW_{-n+2} - sW_{-n+1} - uW_{-n-1}$$

we obtain

$$\begin{aligned}
tnW_{-2n} &= nW_{-2n+3} - rnW_{-2n+2} - snW_{-2n+1} - unW_{-2n-1} \\
t(n-1)W_{-2n+2} &= (n-1) \times W_{-2n+5} - rt(n-1)W_{-2n+4} - st(n-1)W_{-2n+3} - ut(n-1)W_{-2n+1} \\
t(n-2)W_{-2n+4} &= (n-2) \times W_{-2n+7} - r(n-2)W_{-2n+6} - s(n-2)W_{-2n+5} - u(n-2)W_{-2n+3} \\
t(n-3)W_{-2n+6} &= (n-3) \times W_{-2n+9} - r(n-3)W_{-2n+8} - s(n-3)W_{-2n+7} - u(n-3)W_{-2n+5} \\
&\vdots \\
t \times 4 \times W_{-8} &= 3 \times W_{-5} - r \times 4 \times W_{-6} - s \times 4 \times W_{-7} - u \times 4 \times W_{-9} \\
t \times 3 \times W_{-6} &= 3 \times W_{-3} - r \times 3 \times W_{-4} - s \times 3 \times W_{-5} - u \times 3 \times W_{-7} \\
t \times 2 \times W_{-4} &= 2 \times W_{-1} - r \times 2 \times W_{-2} - s \times 2 \times W_{-3} - u \times 2 \times W_{-5} \\
t \times 1 \times W_{-2} &= 1 \times W_{1} - r \times 1 \times W_{0} - s \times 1 \times W_{-1} - u \times 1 \times W_{-3}
\end{aligned}$$

If we add the above equations by side by, we get

$$\begin{aligned} t\sum_{k=1}^{n}kW_{-2k} &= (-(n+1)W_{-2n+1}+1\times W_{1}+\sum_{k=1}^{n}(k+1)W_{-2k+1}) \\ &\quad -r(-(n+1)W_{-2n}+1\times W_{0}+\sum_{k=1}^{n}(k+1)W_{-2k}) \\ &\quad -s(\sum_{k=1}^{n}kW_{-2k+1})-u(nW_{-2n-1}-0\times W_{-1}+\sum_{k=1}^{n}(k-1)W_{-2k+1}). \end{aligned}$$

Since

$$W_{-1}=-\frac{t}{u}W_{0}-\frac{s}{u}W_{1}-\frac{r}{u}W_{2}+\frac{1}{u}W_{3}$$

it follows that

$$\begin{aligned} t\sum_{k=1}^{n}kW_{-2k} &= (-(n+1)W_{-2n+1}+W_{1}+\sum_{k=1}^{n}(k+1)W_{-2k+1}) \\ &\quad -r(-(n+1)W_{-2n}+W_{0}+\sum_{k=1}^{n}(k+1)W_{-2k}) \\ &\quad -s(\sum_{k=1}^{n}kW_{-2k+1})-u(nW_{-2n-1}+\sum_{k=1}^{n}(k-1)W_{-2k+1}). \end{aligned} \tag{3.2}$$

Then, solving system (3.1)-(3.2) (using Theorem 1.2 (b) and (c)), the required result of (b) and (c) follow.

Taking $r=s=t=u=1$ in Theorem 3.1 (a), (b) and (c), we obtain the following proposition.

Proposition 3.1. *If $r=s=t=u=1$ then for $n\geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n}kW_{-k}=\frac{1}{9}(-(3n+1)W_{-n+3}-3W_{-n+2}+(3n-2)W_{-n+1}+(6n+2)W_{-n}+W_{3}+3W_{2}+2W_{1}-2W_{0})$.

(b) $\sum_{k=1}^{n}kW_{-2k}=\frac{1}{9}((3n-10)W_{-2n+2}-(9n-15)W_{-2n+1}+(6n-2)W_{-2n}-(6n-11)W_{-2n-1}-11W_{3}+21W_{2}-4W_{1}+13W_{0})$.

(c) $\sum_{k=1}^{n}kW_{-2k+1}=\frac{1}{9}(-(6n-5)W_{-2n+2}+(9n-12)W_{-2n+1}-(3n-1)W_{-2n}+(3n-10)W_{-2n-1}+10W_{3}-15W_{2}+2W_{1}-11W_{0})$.

From the above proposition, we have the following corollary which gives linear sum formulas of Tetranacci numbers (take $W_n=M_n$ with $M_0=0, M_1=1, M_2=1, M_3=2$).

Corollary 3.2. *For $n\geq 1$, Tetranacci numbers have the following properties.*

(a) $\sum_{k=1}^{n}kM_{-k}=\frac{1}{9}(-(3n+1)M_{-n+3}-3M_{-n+2}+(3n-2)M_{-n+1}+(6n+2)M_{-n}+7)$.

(b) $\sum_{k=1}^{n}kM_{-2k}=\frac{1}{9}((3n-10)M_{-2n+2}-(9n-15)M_{-2n+1}+(6n-2)M_{-2n}-(6n-11)M_{-2n-1}-5)$.

(c) $\sum_{k=1}^{n}kM_{-2k+1}=\frac{1}{9}(-(6n-5)M_{-2n+2}+(9n-12)M_{-2n+1}-(3n-1)M_{-2n}+(3n-10)M_{-2n-1}+7)$.

Taking $W_n=R_n$ with $R_0=4, R_1=1, R_2=3, R_3=7$ in the above proposition, we have the following corollary which presents linear sum formulas of Tetranacci-Lucas numbers.

Corollary 3.3. *For $n\geq 1$, Tetranacci-Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n}kR_{-k}=\frac{1}{9}(-(3n+1)R_{-n+3}-3R_{-n+2}+(3n-2)R_{-n+1}+(6n+2)R_{-n}+10)$.

(b) $\sum_{k=1}^{n} kR_{-2k} = \frac{1}{9}((3n-10)R_{-2n+2} - (9n-15)R_{-2n+1} + (6n-2)R_{-2n} - (6n-11)R_{-2n-1} + 34).$

(c) $\sum_{k=1}^{n} kR_{-2k+1} = \frac{1}{9}(-(6n-5)R_{-2n+2} + (9n-12)R_{-2n+1} - (3n-1)R_{-2n} + (3n-10)R_{-2n-1} - 17).$

Taking $r = 2, s = t = u = 1$ in Theorem 3.1 (a), (b) and (c), we obtain the following proposition.

Proposition 3.2. *If $r = 2, s = t = u = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} kW_{-k} = \frac{1}{16}(-(4n-1)W_{-n+3} + (4n-5)W_{-n+2} + (8n-2)W_{-n+1} + (12n+5)W_{-n} - W_3 + 5W_2 + 2W_1 - 5W_0).$

(b) $\sum_{k=1}^{n} kW_{-2k} = \frac{1}{64}((8n-23)W_{-2n+2} - (40n-51)W_{-2n+1} + (48n-2)W_{-2n} - (24n-29)W_{-2n-1} - 29W_3 + 81W_2 - 22W_1 + 31W_0).$

(c) $\sum_{k=1}^{n} kW_{-2k+1} = \frac{1}{64}(-(24n-5)W_{-2n+2} + (56n-25)W_{-2n+1} - (16n-6)W_{-2n} + (8n-23)W_{-2n-1} + 23W_3 - 51W_2 + 2W_1 - 29W_0).$

From the last proposition, we have the following corollary which gives linear sum formulas of fourth-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5$).

Corollary 3.4. *For $n \geq 1$, fourth-order Pell numbers have the following properties:*

(a) $\sum_{k=1}^{n} kP_{-k} = \frac{1}{16}(-(4n-1)P_{-n+3} + (4n-5)P_{-n+2} + (8n-2)P_{-n+1} + (12n+5)P_{-n} + 7).$

(b) $\sum_{k=1}^{n} kP_{-2k} = \frac{1}{64}((8n-23)P_{-2n+2} - (40n-51)P_{-2n+1} + (48n-2)P_{-2n} - (24n-29)P_{-2n-1} - 5).$

(c) $\sum_{k=1}^{n} kP_{-2k+1} = \frac{1}{64}(-(24n-5)P_{-2n+2} + (56n-25)P_{-2n+1} - (16n-6)P_{-2n} + (8n-23)P_{-2n-1} + 15).$

Taking $W_n = Q_n$ with $Q_0 = 4, Q_1 = 2, Q_2 = 6, Q_3 = 17$ in the last proposition, we have the following corollary which presents linear sum formulas of fourth-order Pell-Lucas numbers.

Corollary 3.5. *For $n \geq 1$, fourth-order Pell-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n} kQ_{-k} = \frac{1}{16}(-(4n-1)Q_{-n+3} + (4n-5)Q_{-n+2} + (8n-2)Q_{-n+1} + (12n+5)Q_{-n} - 3).$

(b) $\sum_{k=1}^{n} kQ_{-2k} = \frac{1}{64}((8n-23)Q_{-2n+2} - (40n-51)Q_{-2n+1} + (48n-2)Q_{-2n} - (24n-29)Q_{-2n-1} + 73).$

(c) $\sum_{k=1}^{n} kQ_{-2k+1} = \frac{1}{64}(-(24n-5)Q_{-2n+2} + (56n-25)Q_{-2n+1} - (16n-6)Q_{-2n} + (8n-23)Q_{-2n-1} - 27).$

If $r = s = t = 1, u = 2$ then $(r+s+t+u-1)(r-s+t-u+1) = 0$ so we can't use Theorem 3.1 (b), (c), directly.

Proposition 3.3. *If $r = s = t = 1, u = 2$ then for $n \geq 1$ we have the following formula:*

$$\sum_{k=1}^{n} kW_{-k} = \frac{1}{16}(-(4n+2)W_{-n+3} - 4W_{-n+2} + (4n-2)W_{-n+1} + (8n+4)W_{-n} + 2W_3 + 4W_2 + 2W_1 - 4W_0).$$

Taking $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1$ in the last proposition, we have the following corollary which presents linear sum formula of fourth-order Jacobsthal numbers.

Corollary 3.6. *For $n \geq 1$, fourth order Jacobsthal numbers have the following property*

$$\sum_{k=1}^{n} kJ_{-k} = \frac{1}{16}(-(4n+2)J_{-n+3} - 4J_{-n+2} + (4n-2)J_{-n+1} + (8n+4)J_{-n} + 8).$$

From the last proposition, we have the following corollary which gives linear sum formulas of fourth order Jacobsthal-Lucas numbers (take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10$).

Corollary 3.7. *For $n \geq 1$, fourth order Jacobsthal-Lucas numbers have the following property*

$$\sum_{k=1}^{n} kj_{-k} = \frac{1}{16}(-(4n+2)j_{-n+3} - 4j_{-n+2} + (4n-2)j_{-n+1} + (8n+4)j_{-n} + 34).$$

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10$ in the last proposition, we have the following corollary which presents linear sum formula of modified fourth order Jacobsthal numbers.

Corollary 3.8. *For $n \geq 0$,modified fourth order Jacobsthal numbers have the following property:*

$$\sum_{k=1}^{n} kK_{-k} = \frac{1}{16}(-(4n+2)K_{-n+3} - 4K_{-n+2} + (4n-2)K_{-n+1} + (8n+4)K_{-n} + 22).$$

From the last proposition, we have the following corollary which gives linear sum formula of fourth-order Jacobsthal Perrin numbers (take $W_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8$).

Corollary 3.9. *For $n \geq 0$, fourth-order Jacobsthal Perrin numbers have the following property:*

$$\sum_{k=1}^{n} kQ_{-k} = \frac{1}{16}(-(4n+2)Q_{-n+3} - 4Q_{-n+2} + (4n-2)Q_{-n+1} + (8n+4)Q_{-n} + 12).$$

Taking $W_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2$ in the last proposition, we have the following corollary which presents linear sum formula of adjusted fourth-order Jacobsthal numbers.

Corollary 3.10. *For $n \geq 0$, adjusted fourth-order Jacobsthal numbers have the following property:*

$$\sum_{k=1}^{n} kS_{-k} = \frac{1}{16}(-(4n+2)S_{-n+3} - 4S_{-n+2} + (4n-2)S_{-n+1} + (8n+4)S_{-n} + 10).$$

From the last proposition, we have the following corollary which gives linear sum formula of modified fourth-order Jacobsthal-Lucas numbers (take $W_n = R_n$ with $R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7$).

Corollary 3.11. *For $n \geq 0$, modified fourth-order Jacobsthal-Lucas numbers have the following property:*

$$\sum_{k=1}^{n} kR_{-k} = \frac{1}{16}(-(4n+2)R_{-n+3} - 4R_{-n+2} + (4n-2)R_{-n+1} + (8n+4)R_{-n} + 12).$$

Taking $r = 2, s = 3, t = 5, u = 7$ in Theorem 3.1 (a), (b) and (c), we obtain the following proposition.

Proposition 3.4. *If $r = 2, s = 3, t = 5, u = 7$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=1}^{n} kW_{-k} = \frac{1}{256}(-(16n+3)W_{-n+3} + (16n-13)W_{-n+2} + (64n+12)W_{-n+1} + (144n+91)W_{-n} + 3W_3 + 13W_2 - 12W_1 - 91W_0)$.

(b) $\sum_{k=1}^{n} kW_{-2k} = \frac{1}{1024}(-(288n+851)W_{-2n+2} + (800n+2627)W_{-2n+1} + (128n-436)W_{-2n} + 7(224n+701)W_{-2n-1} - 701W_3 + 2253W_2 - 524W_1 + 3941W_0)$.

(c) $\sum_{k=1}^{n} kW_{-2k+1} = \frac{1}{1024}((224n+925)W_{-2n+2} - (736n+2989)W_{-2n+1} + (128n+652)W_{-2n} - 7(288n+851)W_{-2n-1} + 851W_3 - 2627W_2 + 436W_1 - 4907W_0)$.

From the last proposition, we have the following corollary which gives linear sum formulas of 4-primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 1, G_3 = 2$).

Corollary 3.12. *For $n \geq 0$, 4-primes numbers have the following properties:*

(a) $\sum_{k=1}^{n} kG_{-k} = \frac{1}{256}(-(16n+3)G_{-n+3} + (16n-13)G_{-n+2} + (64n+12)G_{-n+1} + (144n+91)G_{-n} + 19).$

(b) $\sum_{k=1}^{n} kG_{-2k} = \frac{1}{1024}(-(288n+851)G_{-2n+2} + (800n+2627)G_{-2n+1} + (128n-436)G_{-2n} + 7(224n+701)G_{-2n-1} + 851).$

(c) $\sum_{k=1}^{n} kG_{-2k+1} = \frac{1}{1024}((224n+925)G_{-2n+2} - (736n+2989)G_{-2n+1} + (128n+652)G_{-2n} - 7(288n+851)G_{-2n-1} - 925).$

Taking $W_n = H_n$ with $H_0 = 4, H_1 = 2, H_2 = 10, H_3 = 41$ in the last proposition, we have the following corollary which presents linear sum formulas of Lucas 4-primes numbers.

Corollary 3.13. *For $n \geq 0$, Lucas 4-primes numbers have the following properties:*

(a) $\sum_{k=1}^{n} kH_{-k} = \frac{1}{256}(-(16n+3)H_{-n+3} + (16n-13)H_{-n+2} + (64n+12)H_{-n+1} + (144n+91)H_{-n} - 135).$

(b) $\sum_{k=1}^{n} kH_{-2k} = \frac{1}{1024}(-(288n+851)H_{-2n+2} + (800n+2627)H_{-2n+1} + (128n-436)H_{-2n} + 7(224n+701)H_{-2n-1} + 8505).$

(c) $\sum_{k=1}^{n} kH_{-2k+1} = \frac{1}{1024}((224n+925)H_{-2n+2} - (736n+2989)H_{-2n+1} + (128n+652)H_{-2n} - 7(288n+851)H_{-2n-1} - 10135).$

From the last proposition, we have the following corollary which gives linear sum formulas of modified 4-primes numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 1, E_3 = 1$).

Corollary 3.14. *For $n \geq 0$, modified 4-primes numbers have the following properties:*

(a) $\sum_{k=1}^{n} kE_{-k} = \frac{1}{256}(-(16n+3)E_{-n+3} + (16n-13)E_{-n+2} + (64n+12)E_{-n+1} + (144n+91)E_{-n} + 16).$

(b) $\sum_{k=1}^{n} kE_{-2k} = \frac{1}{1024}(-(288n+851)E_{-2n+2} + (800n+2627)E_{-2n+1} + (128n-436)E_{-2n} + 7(224n+701)E_{-2n-1} + 1552).$

(c) $\sum_{k=1}^{n} kE_{-2k+1} = \frac{1}{1024}((224n+925)E_{-2n+2} - (736n+2989)E_{-2n+1} + (128n+652)E_{-2n} - 7(288n+851)E_{-2n-1} - 1776).$

4 CONCLUSION

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. In this work, sum identities were proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written sum identities in terms of the generalized Tetranacci sequence, and then we have presented the formulas as special cases the corresponding identity for the Tetranacci, Tetranacci-Lucas, fourth-order Pell and other fourth-order linear recurrence sequences. All the listed identities in the corollaries may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.

Computations of the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the genera-lized m-step Fibonacci sequences require the sum of the numbers of the sequences. So, our results can be used to study r-circulant matrices with fourth-order linear recurrence sequences.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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