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Note on Algebras that are Sums of Two Subalgebras Satisfying Polynomial Identities

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Abstract

We study an associative algebra A over an arbitrary field, that is a sum of two subalgebras B and C (i.e. $A = B+C$). We prove that if B has a nil ideal of bounded index, and that C has a commutative ideal, both of finite codimension in B and C, respectively, then for some nil **PI** ideal I of A the ring A/I has a commutative ideal of finite codimension.

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1 Introduction

Let R be an associative ring and R_1, R_2 its subrings such that $R = R_1 + R_2$, i.e. for every $r \in R$ there exist $r_1 \in R_1$ and $r_2 \in R_2$ such that $r = r_1 + r_2$ (we keep this notation throughout the Introduction). In [1] K. I. Beidar and A. V. Mikhalev stated the following problem: if both R_i satisfy polynomial identities (shortly, are **PI** rings), is it true that also R is a **PI** ring? The problem is still open. A positive answer

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in particular cases can be found in many papers (cf. [2], [3], [4], [5], [6], [7], [8], [9]). In the context of these studies the following natural question arises: what properties has the ring R when R_i satisfies a specified polynomial identity $f_i = 0$ for $i = 1, 2$. Concerning this question, Kegel [3] proved that if both R_i are nilpotent then so is R. In [4] it was shown that if R_i are nil of bounded index (i.e. they satisfy identity $x^{n_i} = 0$) then so is R. In [2] Bahturin and Giambruno proved that if both R_i are commutative then R satisfies the identity $[x_1, y_1][x_2, y_2] = 0$, where as usual $[x, y] = xy - yx$.

In [10], developing some ideas of [9], certain generalizations of the above three cited results were obtained for algebras over an arbitrary field that have commutative ideals of finite codimension (such algebras are called almost commutative), algebras that have nilpotent ideals of finite codimension (such algebras are called almost nilpotent) and algebras that have nil ideals of bounded index and finite codimension (called almost nil of bounded index algebras). It was shown in [10] that if both R_i are almost commutative subalgebras then $R = R_1 + R_2$ contains a nilpotent ideal I such that R/I is almost commutative. Moreover, it has been shown that if both R_i are almost nilpotent (respectively almost nil of bounded index) then so is R.

In this paper we show that if R_1 is nil of bounded index and R_2 is commutative then R contains a nil **PI** ideal I such that R/I is commutative. Moreover we show that if R_1 is almost nil of bounded index and R_2 is almost commutative then R contains a nil **PI** ideal I such that R/I is almost commutative.

2 The Main Results

We consider associative algebras over a fixed field K, which are not assumed to have an identity. If I is a two-sided ideal (left ideal, right ideal) of a ring (an algebra) A, we write $I \lhd A$ ($I <_l A$, $I <_r A$).

By $\mathcal{F}$, $\mathcal{N}$, $\mathcal{B}$, $\mathcal{H}$ and $\mathcal{C}$ we denote the class of all finite dimensional algebras, nilpotent algebras, nil of bounded index algebras, nil **PI** algebras, and commutative algebras, respectively.

Let us consider two arbitrary classes of algebras $\mathcal{S}$ and $\mathcal{T}$, for which $0 \in \mathcal{S}$ and $0 \in \mathcal{T}$. Let

$$\mathcal{S}\mathcal{T} = \{A \,|\, \text{there exists } I \lhd A \text{ such that } I \in \mathcal{S} \text{ and } A/I \in \mathcal{T}\}.$$

Obviously $\mathcal{S} \subseteq \mathcal{S}\mathcal{T}$ and $\mathcal{T} \subseteq \mathcal{S}\mathcal{T}$. Thus $\mathcal{C}\mathcal{F}$ denotes the class of almost commutative algebras; $\mathcal{N}\mathcal{F}$ the class of almost nilpotent algebras; $\mathcal{B}\mathcal{F}$ the class of almost nil of bounded index algebras. By Adrunakievich's lemma, if $J \lhd I \lhd A$ and J_A is the ideal of the algebra A generated by J then $(J_A)^3 \subseteq J$. So if $J \in \mathcal{H}$ then $J_A \in \mathcal{H}$. Thus $(\mathcal{H}\mathcal{R})\mathcal{S} = \mathcal{H}(\mathcal{R}\mathcal{S})$ for arbitrary classes of algebras $\mathcal{R}$ and $\mathcal{S}$. Certainly, every subalgebra of an algebra from $\mathcal{H}\mathcal{C}\mathcal{F}$ belongs to $\mathcal{H}\mathcal{C}\mathcal{F}$.

Throughout the paper, A is an algebra over a field K and B and C are subalgebras of A such that $A = B + C$. Moreover, let $B_0 \lhd B$ and $C_0 \lhd C$, where $\dim_K B/B_0 < \infty$ and $\dim_K C/C_0 < \infty$.

Using the above notation, we can state the main result of this paper as follows:

Theorem 2.1. *Assume $A = B + C$ with $B \in \mathcal{B}\mathcal{F}$ and $C \in \mathcal{C}\mathcal{F}$ then $A \in \mathcal{H}\mathcal{C}\mathcal{F}$.*

The proof will be given in Section 4.

3 Preliminary Materials

The centre of an algebra H is denoted by $Z(H)$. For a given subset S of an algebra H, by $l_H(S)$ and $r_H(S)$ we denote the left and right annihilators of S in H, respectively.

We now present several facts which we will use further. Let R be a ring and $I \lhd R$. Applying the identity $[xy, t] = x[y, t] + [x, t]y$, it is easy to show that

1. $I[R, R] \subseteq [I, R]R^*$;
2. $I[I, R] \subseteq [I, I]R^*$,

where R^* denotes the ring R with an identity adjoined. From the above it follows that if I is a commutative ideal of the ring R then $[I,R] \subseteq r_R(I)$ (similar arguments give $[I,R] \subseteq l_R(I)$). Moreover, if $r_I(I) = 0$ or $l_I(I)$=0 then $I \subseteq Z(R)$. Furthermore if $r_R(I) = 0$ or $l_R(I) = 0$ then R is a commutative ring.

We have the following from A. Mekey in [11]:

Lemma 3.1. *Let H be an algebra over an arbitrary field F and P a subalgebra of H such that $\dim_F H/P < \infty$. Then P contains an ideal I of H such that $\dim_F H/I < \infty$.*

We will use the following modification of Petravchuk's Lemma 7 from [9].

Lemma 3.2. *([10]) Let P_1 and P_2 be subalgebras of an algebra H and let I be an ideal of H such that $I \subseteq P_1 + P_2$. Then there exist subalgebras $Q_1 \subseteq P_1$ and $Q_2 \subseteq P_2$ of H such that $Q_1 + Q_2$ is a subalgebra of H and $I \subseteq Q_1 + Q_2$.*

We need some information about the class of nil **PI** rings. Given a ring H, let us denote by $W(H)$ the sum of all nilpotent ideals of the ring H.

Proposition 3.1. *([12]) For every nil **PI** ring H there exists a natural number n such that $H^n \subseteq W(H)$.*

Now we give some properties of the class $\mathcal{HCF}$ (cf. [9, Proposition 1]).

Proposition 3.2. *For the class $\mathcal{HCF}$ the following statements hold:*

(i) every subalgebra and every quotient algebra of an algebra from $\mathcal{HCF}$ belongs to $\mathcal{HCF}$.

(ii) if $P, Q \in \mathcal{HCF}$ then the direct product $P \times Q$ belongs to $\mathcal{HCF}$.

(iii) if $I \lhd H$, $H/I \in \mathcal{HCF}$ and $I \in \mathcal{HCF}$ then $H \in \mathcal{HCF}$.

Proof. The statements (i) and (ii) are obvious. We show that (iii) holds. Let $I \lhd H$, $H/I \in \mathcal{HCF}$ and $I \in \mathcal{HCF}$. Then there exists an ideal J of I and an ideal U of J such that $U \in \mathcal{H}$, $J \in \mathcal{HC}$ and $I/J \in \mathcal{F}$. By Andrunakievich's lemma and the fact that $U \lhd J \lhd I \lhd H$, the ideal $\overline{U}$ of H generated by U is in $\mathcal{H}$, so we can assume that $U = 0$, and thus $I \in \mathcal{CF}$. Let S/I be an ideal of H/I such that $S/I \in \mathcal{HC}$ and $H/S \in \mathcal{F}$. We proceed by induction on $n = \dim I/J$.

Suppose first that $n = 0$, that is $I = J$. Since $[I,H] \subseteq r_I(I) \lhd I \lhd H$ and $(r_I(I))^2 = 0$, $[I,H]$ is contained in some nilpotent ideal of H. So we can assume that $[I,H] = 0$. Hence $I \subseteq Z(H)$ and $I[H,H] = 0$. Let $G = r_S(I)$. Obviously $G \lhd H$ and $[S,S] \subseteq G$. Since $(G \cap I)^2 = 0$ and $G/(G \cap I) \approx (G+I)/I \in \mathcal{BC}$, it follows that $G \in \mathcal{BC}$. We can assume that $G \in \mathcal{C}$. Because $[G,S] \subseteq r_G(G) \lhd G \lhd H$ and $(r_G(G))^2 = 0$ we may assume that $[G,S] = 0$. Therefore $G \subseteq Z(S)$ and $G[S,S] = 0$. Since $[S,S] \subseteq G$, we have $[S,S][S,S] = 0$. It is not hard to check that $S^*[S,S] \lhd S$. Indeed, since $[xy,t] = x[y,t] + [x,t]y$ for all $x,y,z \in S$, we have $S^*[S,S] = [S,S] + S[S,S] \subseteq [S,S] + [S,S]S = [S,S]S^*$. Similarly $[S,S]S^* \subseteq S^*[S,S]$. So $[S,S]S^* = S^*[S,S]$. Hence $S \in \mathcal{NC}$ and consequently $H \in \mathcal{NCF} \subseteq \mathcal{HCF}$.

Suppose now that $n > 0$ and the result holds for integers less than n. Since J is commutative, we again can assume $J \subseteq Z(I)$. Consider $T = \{t \in S \mid It \subseteq J_S\}$, where J_S is the ideal of S generated by J. Obviously $T \lhd S$. Thus S/T can be considered as a subring of the ring of K-linear endomorphisms of I/J_S. Since $I/J_S \in \mathcal{F}$ it follows that $S/T \in \mathcal{F}$. Similarly, $T/L \in \mathcal{F}$ for $L = \{l \in T \mid lI \subseteq J_S\}$. Clearly $J \subseteq L \cap I$.

If $J \neq L \cap I$, then there exists $0 \neq a \in I \setminus J$ such that $Ia \subseteq J_S$ and $aI \subseteq J_S$. Suppose that $Ia \nsubseteq J$ and let $0 \neq b \in Ia \setminus J$. Since $J \subseteq Z(I)$ and $b \in J_S$ then $Ib \subseteq J$ and $bI \subseteq J$. Analogously one can show that if $aI \nsubseteq J$ then there exists $0 \neq c \in aI \setminus J$ such that $cI \subseteq J$ and $Ic \subseteq J$. Thus, in each case there exists $d \in I \setminus J$ such that $dI \subseteq J$ and $Id \subseteq J$. So the subring N of the ring I generated by d and J is a commutative ideal of I. Certainly $\dim I/N < n$ and the induction assumption gives (iii) in the case when $J \neq L \cap I$.

If $J = L \cap I$ then since $L/J = L/(L \cap I) \approx (L+I)/I \in \mathcal{HC}$, J is commutative and $H/L \in \mathcal{F}$. By Lemma 3.1 we can assume that $L \lhd H$, so (iii) holds by the first part of the proof. $\square$

Remark 3.1. Similarly one can prove that the classes $\mathcal{NCF}$ and $\mathcal{BCF}$ are both closed under extensions.

Definition 3.1. An algebra $A = B + C$ over an arbitrary field K is called an $\mathcal{HCF}$-counter-example if A satisfies the following conditions:

(1) $A \notin \mathcal{HCF}$;

(2) the subalgebras B and C have ideals $B_0 \lhd B$ and $C_0 \lhd C$ such that $B_0 \in \mathcal{B}$, $C_0 \in \mathcal{C}$ and the number $\dim A/(B_0 + C_0)$ is the smallest one for all algebras A such that $A \notin \mathcal{HCF}$;

(3) the algebra A does not have nonzero ideals that lie in the K-subspace $B_0 + C_0$ from condition (2).

Let A be an $\mathcal{HCF}$-counter-example. Suppose that $0 \neq I \lhd A$. Denote $\overline{A} = A/I$, $\overline{B} = (B + I)/I$, $\overline{C} = (C + I)/I$, $\overline{B_0} = (B_0 + I)/I$ and $\overline{C_0} = (C_0 + I)/I$. Clearly $\overline{A} = \overline{B} + \overline{C}$ and $\overline{B}, \overline{C} \in \mathcal{HCF}$. By Definition 3.1, $I \nsubseteq B_0 + C_0$, so $\dim \overline{A}/(\overline{B_0} + \overline{C_0}) < \dim A/(B_0 + C_0)$. Thus $\overline{A} \in \mathcal{HCF}$. Hence for every $0 \neq I \lhd A$ we have $A/I \in \mathcal{HCF}$. Basing on this it is easy to show that A is a semiprime algebra. Indeed, if I is a nonzero nilpotent ideal of A then $A/I \in \mathcal{HCF}$ and consequently $A \in \mathcal{HCF}$, which is a contradiction. We can extend the above result, as the following lemma shows.

Lemma 3.3. *If A is an $\mathcal{HCF}$-counter-example, then A is a prime algebra.*

Proof. Suppose that A is not a prime algebra. Hence there exist nonzero ideals I and J of A such that $IJ = 0$. Since $(I \cap J)^2 = 0$ and A is a $\mathcal{HCF}$-counter-example it must be $I \cap J = 0$. Hence A can be embedded into the product $A/I \times A/J$. Since the class $\mathcal{HCF}$ is closed under finite direct products, we obtain that $A \in \mathcal{HCF}$. However A is an $\mathcal{HCF}$-counter-example, so $A \notin \mathcal{HCF}$, which gives a contradiction. $\square$

In the proof of Theorem 2.1 we shall use the following results from [10] and [7] respectively:

Proposition 3.3. *([10]) Let A be a prime algebra. Assume that $C_0 \subseteq Z(C)$, $r_B(B_0) \neq 0$ and $l_B(B_0) \neq 0$. Then $\dim B_0 < \infty$ or $\dim C_0 < \infty$.*

The prime radical will be denoted by β.

Proposition 3.4. *([7]) If R_1 is a nil* **PI** *ring and R_2 is a* **PI** *ring satisfying identity of degree d, then $R_1^{d-1} \subseteq \beta(R)$.*

4 Proof of Theorem 2.1

The proof of Theorem 2.1 goes in a few steps.

Proposition 4.1. *If $B \in \mathcal{B}$ and $C \in \mathcal{C}$, then $A \in \mathcal{HC}$.*

Proof. Suppose that for all $b \in B$ we have $b^m = 0$ for some positive integer m and C is commutative. From Proposition 3.4 it follows that $B \subseteq \beta(A)$. Let G be an ideal of A generated by B. Clearly $A/G = (C + G)/G$ is a commutative algebra and $G \subseteq Nil(A)$, where $Nil(A)$ denotes the nil radical of A. Let $\mathfrak{R}$ be the class of all algebras H such that $H = H_1 + H_2$, where H_1 is a subalgebra of H satisfying the identity $x^m = 0$ and H_2 is a commutative subalgebra of H. Certainly $A \in \mathfrak{R}$ and $\mathfrak{R}$ is closed under direct products. Since for all $H \in \mathfrak{R}$, $H/Nil(H)$ is commutative, by [13, Proposition 1.6.36] we have that all rings of $\mathfrak{R}$ satisfy a polynomial identity $[x, y]^k = 0$ for some positive integer k. So $Nil(A) \in \mathcal{H}$ and $A/Nil(A)$ is commutative. This proves the proposition. $\square$

The following corollary is an immediate consequence of Proposition 4.1.

Corollary 4.1. *If $B \in \mathcal{B}$ and $C \in \mathcal{C}$, then A satisfies a polynomial identity $[x, y]^k = 0$ for some positive integer k.*

Remark 4.1. It is easy to see that Proposition 4.1 and Corollary 4.1 remain true if in the definitions of the classes $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{H}$, the algebras are replaced by rings.

Suppose that $A = B + C$ is an algebra such that $B_0 \in \mathcal{B}$, $C_0 \in \mathcal{C}$, $A \notin \mathcal{HCF}$ and $\dim A/(B_0 + C_0)$ is the smallest number for which $A \notin \mathcal{HCF}$. Let T be the sum of all ideals of A that are contained in $B_0 + C_0$. By Lemma 3.2, $T \subseteq Q_1 + Q_2$, where $Q_1 + Q_2$ is a subalgebra of A and $Q_1 \in \mathcal{B}$, $Q_2 \in \mathcal{C}$. So, by Proposition 4.1, $(Q_1 + Q_2) \in \mathcal{HC}$ and consequently $T \in \mathcal{HC}$. Additionally Proposition 3.2 gives $A/T \notin \mathcal{HCF}$. Clearly $A/T = (B + T)/T + (C + T)/T$. Now it is easy to see that A/T is an $\mathcal{HCF}$-counter-example. We will use this observation in proofs of Proposition 4.2 and Theorem 2.1.

Proposition 4.2. *If $B \in \mathcal{NF}$ and $C \in \mathcal{CF}$, then $A \in \mathcal{HCF}$.*

Proof. Suppose the result is false. So by the paragraph following Corollary 4.1, we can assume that A is an $\mathcal{HCF}$-counter-example. Let B_0 be nilpotent and C_0 be commutative. Consider $A_1 = B + B_0A$. It is clear that A_1 is a subalgebra of A and since $B \subseteq A_1$, we have $A_1 = A_1 \cap (B + C) = B + A_1 \cap C$. We shall show that $A_1 \in \mathcal{HCF}$. We proceed by induction on n, where n is a positive integer such that $B_0^n = 0$. For $n = 1$, we have $\dim B < \infty$, so $A_1 \in \mathcal{CF}$ by Lemma 3.1. Assume that $n > 2$ and the result holds for all smaller integers. By Lemma 3.3, A is a prime algebra. Since $B_0^n = 0$ we have that $B_0^{n-1} <_l A_1$. From this, there exists a nilpotent ideal I of A_1 such that $B_0^{n-1} \subseteq I$. Certainly $A_1/I = (B + I)/I + ((A_1 \cap C) + I)/I$, where $(B + I)/I \in \mathcal{NF}$, $((A_1 \cap C) + I)/I \in \mathcal{CF}$ and $(B_0 + I)/I$ is a nilpotent ideal of $(B + I)/I$ while $((A_1 \cap C_0) + I)/I$ is a commutative ideal of $((A_1 \cap C) + I)/I$, both of finite codimension in $(B + I)/I$ and $((A_1 \cap C) + I)/I$, respectively. Moreover $((B_0 + I)/I)^{n-1} = 0$, so the induction assumption gives $A_1/I \in \mathcal{HCF}$ and, since I is nilpotent, we have $A_1 \in \mathcal{HCF}$. Let $U_1 = B_0 + B_0A$. Since $U_1 \subseteq A_1$ it follows that $U_1 \in \mathcal{HCF}$.

Consider $A_2 = C + C_0A$. Since $C \subseteq A_2$, it is clear that $A_2 = A_2 \cap (B + C) = C + A_2 \cap B$. Now we shall show that $A_2 \in \mathcal{HCF}$. Suppose that $A_2 \notin \mathcal{HCF}$. Let us note that $\dim A_2/(C_0 + (A_2 \cap B_0)) \leq \dim A/(B_0 + C_0)$, so since $A_2 \notin \mathcal{HCF}$, we obtain $\dim A_1/(C_0 + (A_2 \cap B_0)) = \dim A/(B_0 + C_0)$. Consider A_2/T, where T is the sum of all ideals of A_2 that lie in the K-subspace $C_0 + (A_2 \cap B_0)$. By Proposition 4.1 and Lemma 3.2, $T \in \mathcal{HC}$. If $A_2/T \in \mathcal{HCF}$ then $A_2 \in \mathcal{HCF}$, contrary to the assumption. Hence A_2/T is an $\mathcal{HCF}$-counter-example. It is obvious that $A_2/T = (C + T)/T + ((A_2 \cap B) + T)/T$. Moreover $l_{C_0}(C_0) <_r A_2$. Since $[C, C_0] \subseteq l_{C_0}(C_0)$ and $(l_{C_0}(C_0))^2 = 0$ and A_2/T is a prime algebra, it follows that $[C, C_0] \subseteq T$. Hence $(C_0 + T)/T \subseteq Z((C + T)/T)$. Let $\overline{B_0} = ((A_2 \cap B_0) + T)/T$. If $\overline{B_0} = 0$, then Lemma 3.1 implies that $(C_0 + T)/T$ contains an ideal $S \lhd A_2/T$ of finite codimension, so $A_2/T \in \mathcal{HCF}$. If $\overline{B_0} \neq 0$ then since $\overline{B_0}$ is nilpotent, $l_{\overline{B_0}}(\overline{B_0}) \neq 0$ and $r_{\overline{B_0}}(\overline{B_0}) \neq 0$. Therefore by Proposition 3.3 and Lemma 3.1 again $A_2/T \in \mathcal{HCF}$. We have obtained a contradiction with A_2 being an $\mathcal{HCF}$-counter-example. So indeed $A_2 \in \mathcal{HCF}$. Let $U_2 = C_0 + C_0A$. Since $U_2 \subseteq A_2$, we have $U_2 \in \mathcal{HCF}$.

In particular U_1 and U_2 are **PI** algebras. It is clear that $U_1 + U_2$ is a subalgebra of A, $U_1 <_r A$ and $U_2 <_r A$. By [7, Corollary 4], $U_1 + U_2$ is a **PI** algebra. But $\dim A/(U_1 + U_2) < \infty$, so according to Lemma 3.1, there exists $J \lhd A$ such that $J \subseteq U_1 + U_2$ and $\dim A/J < \infty$. Consequently A is a **PI** algebra. Moreover, if $r_{C_0}(C_0) = 0$ then $C_0 \subseteq Z(C)$ and it is enough to apply Proposition 3.3 and Lemma 3.1 to obtain $A \in \mathcal{NF}$ or $A \in \mathcal{CF}$, contrary to A being an $\mathcal{HCF}$-counter-example. Thus $r_{C_0}(C_0) \neq 0$.

We show now that $Z(A)$ is finite dimensional over K. This will be proved by showing that $Z(A) \cap (B_0 + C_0) = 0$. Suppose, contrary to our claim, that $Z(A) \cap (B_0 + C_0) \neq 0$, so there exists $0 \neq z = b_0 + c_0$, where $b_0 \in B_0$, $c_0 \in C_0$ and $z \in Z(A)$. Certainly $l_{B_0}(B_0)zr_{C_0}(C_0) = 0$. Since A is a prime algebra and $z \in Z(A)$, it follows that $l_{B_0}(B_0)r_{C_0}(C_0) = 0$. But $l_{B_0}(B_0) \lhd B$ and $r_{C_0}(C_0) \lhd C$, so $l_{B_0}(B_0)Ar_{C_0}(C_0) \subseteq l_{B_0}(B_0)Br_{C_0}(C_0) + l_{B_0}(B_0)Cr_{C_0}(C_0) \subseteq l_{B_0}(B_0)r_{C_0}(C_0) = 0$, contrary to the primeness of A. Hence $Z(A) \cap (B_0 + C_0) = 0$. Therefore $\dim_K Z(A) < \infty$.

Now we prove that $\dim_K A < \infty$. Since A is a prime algebra, $Z(A)$ is a commutative finite dimensional domain. It follows that $Z(A)$ is a field. So the central localization $Z(A)^{-1}A$ of A is equal to A. Since A is a **PI** algebra, Posner's Theorem [14] implies that A is finite dimensional over $Z(A)$. Consequently $\dim_K A < \infty$. Thus A is not an $\mathcal{HCF}$-counter-example which gives a contradiction. $\square$

We are now in position to prove our main result.

Proof of Theorem 2.1: Suppose that there exists $A \notin \mathcal{HCF}$. Without loss of generality, we can assume that A is an $\mathcal{HCF}$-counter-example. Moreover, if $B_0 \in \mathcal{B}$ and $C_0 \in \mathcal{C}$ and $\dim A/(B_0 + C_0)$ is the smallest number for which $A \notin \mathcal{HCF}$. Applying Lemma 3.3 we have that A is a prime algebra. By Proposition 3.1, there exists a natural number $n > 0$ such that $B_0^n \subseteq W(B_0)$. Observe that if L and K are ideals of B such that $LK = 0$ then $\overline{B_0} = B_0 + KAL$ is a subalgebra of A, $KAL \lhd \overline{B_0}$ and $(KAL)^2 = 0$. Since $\overline{B_0}/(KAL)$ is a homomorphic image of B_0, it follows that $\overline{B_0}$ is nil of bounded index.

Consider $\overline{B} = B + KAL$. It is clear that $\overline{B}$ is a subalgebra of A, $\overline{B_0} \lhd \overline{B}$ and $\dim \overline{B}/\overline{B_0} < \infty$. Moreover $A = \overline{B} + C$. If $a \in \overline{B_0} \cap C_0$, then $aA = a\overline{B} + aC$. Clearly $a\overline{B}$ is a nil subalgebra of bounded index of aA. Since C_0 is commutative and $a \in \overline{B_0} \cap C_0$, also aC is a nil subalgebra of bounded index of aA. By [4, Theorem 2], $aA \in \beta(A)$. Since A is a prime algebra, $a = 0$. Thus if $B_0 \subsetneq \overline{B_0}$ then $\dim A/(\overline{B_0} + C_0) < \dim A/(B_0 + C_0)$, which contradicts the choice of B_0 and C_0. Hence $\overline{B_0} = B_0$. It follows that $KAL \subseteq B_0$. In particular, if $I \lhd B$, $I^m = 0$ and $I^{m-1} \neq 0$ for a positive integer m, then for every $1 \leq i \leq m-1$, we have $A_i = I^{m-i}AI^i \subseteq B_0$. Since $B_0 \in \mathcal{B}$, B_0 is a **PI** algebra of degree, say, d. So it satisfies the identity

$$x_1 x_2 \dots x_d = \sum_{id \neq \pi \in S_d} \alpha_\pi x_{\pi(1)} x_{\pi(2)} \dots x_{\pi(d)},$$

where S_d is the set of permutations of the set $\{1, 2, \dots d\}$ and α_π are some integers. Suppose that $m > d$. Then

$$(I^{m-1}A)^d I^d = A_1 A_2 \dots A_d = \sum_{id \neq \pi \in S_d} \alpha_\pi A_{\pi(1)} A_{\pi(2)} \dots A_{\pi(d)} = 0,$$

so $(I^{m-1}A)^{d+1} = 0$. Since A is prime, it follows that $I^{m-1} = 0$, which gives a contradiction. Thus $m \leq d$. So for all nilpotent ideals I of B, $I^d = 0$. Since every nilpotent ideal of B_0 generates a nilpotent ideal of B, for every nilpotent ideal J of B_0 we have $J^d = 0$. Hence $(W(B_0))^d = 0$. But $B_0^n \subseteq W(B_0)$, so B_0 is a nilpotent ideal of B. Therefore $B \in \mathcal{NF}$. Using Proposition 4.2 we have $A \in \mathcal{HCF}$. This contradicts our assumption that A is an $\mathcal{HCF}$-counter-example and completes the proof.

5 Conclusions

In the context of the main result of the paper, that is Theorem 2.1, the following problem arises: if $B \in \mathcal{BF}$ and $C \in \mathcal{CF}$, is it true that $A \in \mathcal{BCF}$? Reasoning used in the article can be applied with small modifications to obtain the positive answer to the above question. However, Proposition 4.1 should be enhanced.

Competing Interests

Author has declared that no competing interests exist.

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