Archives of Current Research International
1(1): 17-27, 2014, Article no.ACRI.2014.002

SCIENCEDOMAIN ***international***
www.sciencedomain.org

Optimal Control Problem for a Modified Novikov Equation with Dissipation

Jiangbo Zhou[1*], Jing Chen[1] and Junde Chen[1]

[1]Department of Mathematics, Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu 212013, China.

Authors' contributions

This work was carried out in collaboration between all authors. Authors Jing Chen and JC designed the study, worked out the existence of a unique weak solution the modified Novikov equation. Author JZ obtained the existence of an optimal solution to the optimal control problem. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ACRI/2014/15087
Editor(s):
(1) Yanpeng Li, Research Fellow, Department of Health Science Research, Mayo Clinic, USA.
Reviewers:
(1) Anonymous, Moscow Technological Institute, Russia.
(2) W. Obeng-Denteh, Mathematics Department, Kwame Nkrumah University of Science and Technology, Kumasi, Ghana.
Complete Peer review History: http://www.sciencedomain.org/review-history.php?iid=904&id=41&aid=7605

Original Research Article

Received 5th November 2014
Accepted 22nd November 2014
Published 3rd January 2015

ABSTRACT

This paper investigate the optimal control problem for a modified Novikov equation with dissipation. The existence and uniqueness of regular local solution to the corresponding initial boundary value problem is guaranteed by the Faedo-Galerkin method. A critical estimate of the solution is obtained. Finally, the existence of an optimal solution to the control problem is proved.

Keywords: Modified Novikov equation; optimal control; weak solution.

MSC (2010): 49J20,49J45,35B37,35F30

**Corresponding author: Email: ujszjb@126.com;*

1. INTRODUCTION

The Novikov equation

$$u_t - u_{xxt} + 4u^2u_x = 3uu_xu_{xx} + u^2u_{xxx} \qquad (1.1)$$

was isolated by Novikov [1] in a symmetry classification of nonlocal partial differential equations. Compared with the Camassa-Holm equation [2]

$$u_t - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx} \qquad (1.2)$$

the Novikov equation has nonlinear terms that are cubic, rather than quadratic, which can be thought as a generalization of the Camassa-Holm equation. Novikov [1] found its first few symmetries and he subsequently found a scalar Lax pair for it, proving that the equation is integrable. Hone and Wang [3] gave a matrix Lax pair for the Novikov equation, and showed how it was related by a reciprocal transformation to a negative flow in the Sawada-Kotera hierarchy. Infinitely many conserved quantities were found, as well as a bi-Hamiltonian structure. They also presented peakons for Eq. (1.1). Liu, Liu and Qu [4] proved such peakons are orbital stable. Hone, Lundmark and Szmigielski [5] calculated the explicit formulas for multipeakon solutions of Eq. (1.1), using the matrix Lax pair found by Hone and Wang. Very recent work are intensively devoted to studying the local well-posedness, global existence of strong and weak solution, and blow up of solution of initial value problem for Eq. (1.1) in Sobolev spaces or Besov spaces [6-13].

In general, the energy dissipation mechanisms are difficult to avoid in a real world and many authors considered the dissipative effects on the practical models. For example, Ott and Sudan [14] investigated the KdV equation with the presence of dissipation and their effect on solution of the KdV equation. The long time behavior of solutions to the weakly dissipative KdV equation was studied by Ghidaglia [15]. Recently, Wu and Yin investigated the weakly dissipative Camassa-Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx} - \lambda\left(u - u_{xx}\right) \qquad (1.3)$$

on the line [16] and on the circle [17]. They also studied the weakly dissipative Degasperis-Procesi equation

$$u_t - u_{xxt} + 4uu_x = 3u_xu_{xx} + uu_{xxx} - \lambda\left(u - u_{xx}\right) \qquad (1.4)$$

on the line in [18,19] and on the circle [20], where $\lambda > 0$ is a constant. In [21], Yan, Li and Zhang considered the global existence and blow-up for the weakly dissipative Novikov equation

$$u_t - u_{xxt} + 4u^2u_x = 3uu_xu_{xx} + u^2u_{xxx} - \lambda\left(u - u_{xx}\right)$$
$$\lambda > 0. \qquad (1.5)$$

In this paper, we investigate an optimal control problem governed by a modified Novikov equation with dissipation

$$m_t + u^2m_x + uu_xm - \varepsilon m_{xx} = 0, m = u - u_{xx} \qquad (1.6)$$

where $\varepsilon > 0$ denotes the dissipation. When $\varepsilon = 0$, the equation is one of the modified Novikov equations recently studied by Mi and Mu (see, Eq. (1.1) in [22] with $b = 1$). In comparison with most of the low-order nonlinear equations, such as the Burgers equation [see, 23,24,25], the Camassa-Holm equation [2,26], the Degasperis-Procesi equation [27] and the Dullin-Gottwalld-Holm equation [28], the Novikov equation has higher order nonlinear terms. Therefore, the treatment of this equation is generally of more difficulty and challenge. One can see this point in the study of the Cauchy problem of this equation (see, for instance, [6-13]).

The optimal control is an important component of modern control theories and has a wider application in modern engineering. Two methods are introduced to study the control problems in partial differential equation (PDE): One is using a low model method, and then changing to an ordinary differential equation (ODE) model [29]; the other is using a quasi-optimal control method [30]. No matter which one is chosen, it is necessary to prove the existence of optimal solution according to the basic theory [31]. Let us mention some papers concerning optimal control problems for nonlinear PDEs. Ghattas and Bark [32] studied the optimal control of two and three dimensional incompressible Navier-Stokes flows. Vedantham [23] developed a technique to utilize the Cole-Hopf transformation to solve an optimal control problem for the Burgers equation. Sang-Uk Ryu and Atsushi Yagi [33] investigated the optimal control of the Keller-Segel equations. Volkwein [24] used the augmented Lagrangian-

SQP technique to solve the optimal control problem governed by the Burgers equaiton. Hinze and Volkwein [25] discussed the instantaneous control of the Burgers equation. Lagnese and Leugering [34] considered the problem of boundary optimal control of a wave equation with boundary dissipation. Oksendal [35] proved a sufficient maximum principle for the optimal control systems described by a quasi-linear stochastic heat equation. Based on the energy estimates and the compact method, Tian, Shen et al. [26-28] studied the optimal control problems for the viscous Camassa-Holm equation, viscous Degasperis-Procesi equation and viscous Dullin-Gottwalld-Holm equation. Under boundary condition, Zhao and Liu [36] studied optimal control problem for viscous Cahn-Hilliard equation.

Our paper is organized as follows. In Section 2, we formulate the control problem. In Section 3, we obtain the existence of a unique regular solution to the dissipative modified Novikov equation and a critical estimate of this equation. In Section 4, we prove the existence of an optimal solution to the control problem. A short conclusion is made in Section 5.

2. FORMULATION OF THE OPTIMAL CONTROL PROBLEM

It is appropriate to introduce some notations. Let Ω denote the interval $(0,1)$, and Ω_0 be a subset of Ω with positive measure. Define

$$H \triangleq \dot{L}^2(\Omega) = \left\{ u(x) = \sum_{k\in Z} u_k \exp(2i\pi kx) : \sum_{k\in Z} |u_k|^2 < \infty, \int_\Omega u(x)dx = 0 \right\}$$

which is endowed with the scalar product.

$$(u,v)_H = \int_\Omega u(x)\cdot v(x)\,dx,$$

and the norm

$$\|u\|_{L^2(\Omega)} = \left((u,u)_H\right)^{\frac{1}{2}}$$

Let $L^\infty(\Omega)$ be the space of real functions on Ω which are measurable and essentially bounded; it is a Banach for the norm

$$\|u\|_{L^\infty(\Omega)} = \sup_{x\in\Omega} ess\,|u(x)| \ .$$

We introduce the Sobolev space

$$H^s(\Omega) = \left\{ u : u \in L^2(\Omega), D^j u \in L^2(\Omega), j = 1,2,\cdots,s \right\} \ , \ s \in N$$

which is equipped with the scalar product

$$(u,v)_{H^s(\Omega)} = \sum_{j=0} \left(D^j u, D^j v\right)_{L^2(\Omega)} \ ,$$

with the norm

$$\|u\|_{H^s(\Omega)} = \left((u,u)_{H^s(\Omega)}\right)^{\frac{1}{2}}.$$

Set

$$H^s_{per} = \left\{ u(x) = \sum_{k\in Z} u_k \exp(2i\pi kx) : \sum_{k\in Z} \left(1+|k|^2\right)^s |u_k|^2 < \infty, \int_\Omega u(x)dx = 0 \right\}$$

We have on $V \overset{\Delta}{=} H^1_{per}$ a Poincaré's inequality

$$\|u\|_{L^2(\Omega)} \le c_0(\Omega)\|Du\|_{L^2(\Omega)} \ , \ \forall u \in V \ .$$

This shows that V is Hilbertian for the scalar product $(\varphi,\psi)_V = (\varphi',\psi')_H$ and $\|u\|_V = \left((u,u)_V\right)^{\frac{1}{2}}$ is a norm on this space equivalent to that induced by $H^1(\Omega)$, see [37].

Denote the dual spaces of V and H as $V^* = H^{-1}(\Omega)$ and H^*, respectively. Obviously, $V \to H = H^* \to V^*$, each embedding being dense. The duality pairing between V^* and V is given by

$(u,\varphi)_{V^*,V} = \int u(x)\cdot v(x)\,dx$ for $u \in V^*$ and $\varphi \in V$.

For given $T > 0$ and Banach space X , let $L^2(0,T;X)$ and $C(0,T;X)$ denote the space of square integrable and continuous functions, respectively, in the sense of Bochner from $[0,T]$ to X . They are Banach spaces for the norms

$$\|f\|_{L^2(0,T;X)} = \left(\int_0^T \|f(t)\|_X^2\, dt\right)^{\frac{1}{2}} ,$$

and

$$\|f\|_{C(0,T;X)} = \sup_{t\in[0,T]} \|f(t)\|_X ,$$

respectively. $L^\infty(0,T;X)$ denotes the space of measurable functions from $(0,T)$ to X which are essentially bounded; the space is Banach for the norm

$$\|u\|_{L^\infty(0,T;X)} = \sup_{t\in(0,T)} ess \|f(t)\|_X ,$$

For fixed $T>0$, we also defined the space $W(0,T;V)$ as

$$W(0,T;V) = \left\{m : m\in L^2(0,T;V), m_t \in L^2(0,T;V^*)\right\} ,$$

which is a Hilbert space endowed with the common inner product, see [38], the corresponding norm being denoted $\|\varphi\|_{W(0,T;V)}$,

$$\|\varphi\|_{W(0,T;V)} = \left(\|\varphi\|_{L^2(0,T;V)} + \|\varphi_t\|_{L^2(0,T;V^*)}\right)^{\frac{1}{2}} .$$

For brevity we write $L^2(X)$, $C(X)$ and $W(V)$ in place of $L^2(0,T;X)$, $C(0,T;X)$ and $W(0,T;X)$.

Further, the extension operator $B^* \in \mathrm{L}\left(L^2(Q_0), L^2(V^*)\right)$ is defined by

$$B^* q = \begin{cases} q, & in \quad Q_0\, Q_0 \\ 0, & in \quad Q\setminus Q_0 \end{cases}$$

where $Q=(0,T)\times\Omega$ and $Q_0=(0,T)\times\Omega_0$.

The optimal control problem we intend to investigate is minimizing

$$J(m,\varpi) = \frac{1}{2}\|Cm - z\|_{L^2(H)}^2 + \frac{\delta}{2}\|\varpi\|_{L^2(Q_0)}^2 \quad (2.1)$$

subject to

$$\begin{cases} m_t - \varepsilon m_{xx} + 3uu_x m + u^2 m_x = B^*\varpi \quad in \quad Q \\ \dfrac{\partial^j u}{\partial x^j}(0,t) = \dfrac{\partial^j u}{\partial x^j}(1,t), \quad j=0,1,2, \\ m(0,x) = m_0(x) \quad in \quad \Omega \end{cases} \quad (2.2)$$

Here C is the injection of $W(V)$ into $L^2(H)$, z is a desired state, $\delta>0$ is fixed, $m_0(x)\in H$ is given, and $L^2(Q_0)$ denotes the control variable. Clearly, our control target is to match the given desired state z by adjusting the body force ϖ in a control volume $Q_0\subseteq Q$ in the L^2-sense.

For a control $\varpi\in L^2(Q_0)$ the state $m\in W(V)$ is given by the weak solution of the viscous Novikov equation

$$\begin{cases} m_t - \varepsilon m_{xx} + uu_x m + u^2 m_x = B^*\varpi \quad in \quad L^2(V^*) \\ m(0,x) = m_0(x) \quad in \quad H \end{cases} \quad (2.3)$$

Now we present the definition of the weak solution to Eq. (2.3) in the space $W(V)$.

Definition 2.1 A function $m(x,t)\in W(V)$ is called a weak solution to Eq. (2.3), if

$$(m_t,\phi)_{V^*,V} + \varepsilon(m_t,\phi)_V + (uu_x m + u^2 m_x,\phi)_{V^*,V} = (B^*\varpi,\phi)_{V^*,V} \quad (2.4)$$

is valid for all $\phi\in V$ and a.e. $t\in[0,T]$ and $m_0\in H$ are valid.

The control problem then is

$$\min J(m,\varpi) \quad \text{s.t.} \quad (m,\varpi) \text{ is a weak solution to } (2.3) \quad (2.5)$$

Let $X=W(V)\times L^2(Q_0)$, $Y=L^2(V)\times H$, and define an operator $e=e(e_1,e_2): X\to Y$ by

$$e_1 = (-\Delta)^{-1}\left(m_t - \varepsilon m_{xx} + uu_x m + u^2 m_x - B^*\varpi\right)$$

and

$$e_2 = m(x,0) - m_0(x) ,$$

where $\Delta : V \to V^*$ is the Laplace operator with periodic boundary conditions. Then we rewrite (2.5) in the following form

$$\min J(m,\varpi) \quad \text{s.t.} \quad e(m,\varpi)=0 \qquad (2.6)$$

3. EXISTENCE OF A UNIQUE WEAK SOLUTION TO (2.3)

The following theorem ensures the existence of a unique weak solution to Eq. (2.3)

Theorem 3.1 Let $T>0$ be fixed, $m_0(x)\in H$ and $B^*\varpi \in L^2(V^*)$, then there exists a unique local weak solution $m(x,t)\in C(H)\cap W(V)$ to Eq. (2.3) in the sense of Definition 2.1. Moreover, there exists a constant $C>0$, which depends on ε, ϖ and m_0, such that

$$\|m\|_{W(V)} \le C\left(1+\|m_0\|_H+\|m_0\|_H^2+\|\varpi\|_{L^2(Q_0)}+\|\varpi\|_{L^2(Q_0)}^2\right) \quad (3.1)$$

Proof. As in the proof for the unsteady Navier-Stokes equations in [39] we derive that there exists a unique $W(V)$ satisfying (2.3) in the weak sense. Thus, we have to prove the estimate (3.1)

Multiplying the first equation in (3.2) by $2m$ and integrating over Ω, we obtain that

$$\frac{d}{dt}\|m\|_H^2 + 2\varepsilon\|m\|_V^2 = 2\left(B^*\varpi, m\right)_{V^*,V} \qquad (3.2)$$

Applying Hölder's inequality and Young's inequality for products, we obtain

$$\left|2\left(B^*\varpi, m\right)_{V^*,V}\right| \le 2\|B^*\varpi\|_{V^*}\|m\|_V$$

$$\le \frac{4}{\varepsilon}\|B^*\varpi\|_{V^*}^2 + \varepsilon\|m\|_V^2 \qquad (3.3)$$

Then we derive from (3.2)-(3.3) that

$$\frac{d}{dt}\|m\|_H^2 + \varepsilon\|m\|_V^2 \le \frac{4}{\varepsilon}\|B^*\varpi\|_{V^*}^2 . \qquad (3.4)$$

Integrating it over $[0,t)$, we have

$$\begin{aligned}\|m(t)\|_H^2 &\le \|m_0\|_H^2 + \frac{4}{\varepsilon}\int_0^t \|B^*\varpi(s)\|_{V^*}^2 \\ &\le \|m_0\|_H^2 + \frac{4}{\varepsilon}\int_0^T \|B^*\varpi(s)\|_{V^*}^2\, ds \\ &= \|m_0\|_H^2 + \frac{4}{\varepsilon}\|B^*\varpi\|_{L^2(V^*)}^2, \end{aligned} \qquad (3.5)$$

which implies that

$$\|m(t)\|_{C(H)}^2 \le \|m_0\|_H^2 + \frac{4}{\varepsilon}\|B^*\varpi\|_{L^2(V^*)}^2 . \qquad (3.6)$$

Meanwhile, it follows from (3.4) that

$$\begin{aligned}\|m\|_{L^2(V)}^2 &= \int_0^T \|m(t)\|_V^2\, dt \\ &\le \frac{4}{\varepsilon^2}\int_0^T \|B^*\varpi(s)\|_{L^2(V^*)}^2 ds + \frac{1}{\varepsilon}\|m_0\|_H^2 \\ &= \frac{4}{\varepsilon^2}\|B^*\varpi\|_{L^2(V^*)}^2 + \frac{1}{\varepsilon}\|m_0\|_H^2 . \end{aligned} \qquad (3.7)$$

Since

$$\begin{aligned}\|m\|_H^2 &= \|u-u_{xx}\|_H^2 = \left(u-u_{xx}, u-u_{xx}\right)_H \\ &= \|u\|_H^2 + 2\|u_x\|_H^2 + \|u_{xx}\|_H^2 \\ &= \|u\|_H^2 + 2\|u\|_V^2 + \|u_{xx}\|_H^2 , \end{aligned} \qquad (3.8)$$

and

$$\|u_x\|_{H^1}^2 = \|u_x\|_H^2 + \|u_{xx}\|_H^2 , \qquad (3.9)$$

we obtain that

$$\|u\|_H \le \|m\|_H \le \|m\|_{C(H)},$$
$$\|u\|_V \le \|m\|_H \le \|m\|_{C(H)}, \qquad (3.10)$$
$$\|u_x\|_{H^1} \le \|m\|_H \le \|m\|_{C(H)}.$$

Using Hölder's inequality, Poincaré's inequality and the Sobolev embedding theorem, we deduce from the first equation in (3.2) that

$$\|m_t\|_{L^2(V^*)}$$
$$= \sup_{\|\varphi\|_{L^2(V)}} \left| \int_0^T \int_\Omega \left(\varepsilon m_{xx}\varphi - u u_x m\varphi - u^2 m_x \varphi - B^*\varpi\varphi \right) dxdt \right|$$
$$= \sup_{\|\varphi\|_{L^2(V)}} \left| \int_0^T \int_\Omega \left(\varepsilon m_{xx}\varphi - \frac{1}{2} u^2 m_x \varphi + \frac{1}{2} u^2 m\varphi_x + B^*\varpi\varphi \right) dxdt \right|$$
$$\le \sup_{\|\varphi\|_{L^2(V)}} \int_0^T \left(\varepsilon \|m_x\|_H \|\varphi_x\|_H + \frac{1}{2}\|u\|_{L^\infty}^2 \|m_x\|_H \|\varphi\|_H + \frac{1}{2}\|u\|_{L^\infty}^2 \|m\|_H \|\varphi_x\|_H + \|B^*\varpi\|_{V^*} \|\varphi\|_H \right) dt$$
$$\le \sup_{\|\varphi\|_{L^2(V)}} \int_0^T \left(\varepsilon \|m\|_V \|\varphi\|_V + \|u\|_V^2 \|m\|_V \|\varphi\|_V + \|u\|_V^2 \|m\|_V \|\varphi\|_V + \|B^*\varpi\|_{V^*} \|\varphi\|_V \right) dt \qquad (3.11)$$
$$\le \sup_{\|\varphi\|_{L^2(V)}} \int_0^T \left(\varepsilon \|m\|_V \|\varphi\|_V + \|m\|_H^2 \|m\|_V \|\varphi\|_V + \|B^*\varpi\|_{V^*} \|\varphi\|_V \right) dt$$
$$\le \left(\varepsilon + 2\|m\|_{C(H)} \|m\|_{L^2(V)} + \|B^*\varpi\|_{L^2(V^*)} \right)$$

where we used the fact that $\|u(t)\|_V \le \|m\|_{C(H)}$ in (3.10).

It then follows from (3.6), (3.7) and (3.11) that

$$\|m\|_{W(V)}^2 = \|m\|_{L^2(V)}^2 + \|m_t\|_{L^2(V^*)}^2$$
$$\le \|m\|_{L^2(V)}^2 + \left(\left(\varepsilon + 2\|m\|_{C(H)} \right) \|m\|_{L^2(V)} + \|B^*\varpi\|_{L^2(V^*)} \right)^2$$
$$\le \|m\|_{L^2(V)}^2 + \left(\varepsilon + 2\|m\|_{C(H)} \right)^2 \|m\|_{L^2(V)}^2 + \|\varpi\|_{L^2(Q_0)}^2 + 2\left(\varepsilon + 2\|m\|_{C(H)} \right) \|m\|_{L^2(V)} \|\varpi\|_{L^2(Q_0)}$$
$$\le \|m\|_{L^2(V)}^2 + 2\left(\varepsilon^2 + 4\|m\|_{C(H)}^2 \right) \|m\|_{L^2(V)}^2 + \|\varpi\|_{L^2(Q_0)}^2 + \varepsilon\left(\|m\|_{L^2(V)}^2 \right) + \|\varpi\|_{L^2(Q_0)}^2 + 2\left(\|m\|_{C(H)}^2 + \|m\|_{L^2(V)}^2 \right) \|\varpi\|_{L^2(Q_0)}^2 \qquad (3.12)$$
$$\le \left[1 + 2\left(\varepsilon^2 + 4\|m\|_{C(H)}^2 \right) + \|\varpi\|_{L^2(Q_0)} \right] \|m\|_{L^2(V)}^2 + (\varepsilon+1)\|\varpi\|_{L^2(Q_0)}^2 + 2\|\varpi\|_{L^2(Q_0)} \|m\|_{C(H)}^2$$
$$\le \left(1 + 2\left(\varepsilon^2 + 4\|m_0\|_H^2 + \frac{16}{\varepsilon}\|\varpi\|_{L^2(Q_0)}^2 \right) + \|\varpi\|_{L^2(Q_0)} \right)$$
$$\times \left(\frac{4}{\varepsilon^2}\|\varpi\|_{L^2(Q_0)}^2 + \frac{1}{\varepsilon}\|m_0\|_H^2 \right) + (\varepsilon+1)\|\varpi\|_{L^2(Q_0)}^2 + 2\|\varpi\|_{L^2(Q_0)} \left(\|m_0\|_H^2 + \frac{4}{\varepsilon}\|\varpi\|_{L^2(Q_0)} \right)$$
$$\le C\left(1 + \|m_0\|_H + \|m_0\|_H^2 + \|\varpi\|_{L^2(Q_0)} + \|\varpi\|_{L^2(Q_0)}^2 \right)^2$$

which gives the claim (3.1).

4. EXISTENCE OF AN OPTIMAL SOLUTION TO (2.6)

In this section, we prove the existence of an optimal solution to the control problem (2.6) based on Lions' theory (see [40]).

Theorem 4.1 There exists an optimal control solution (m^*, ϖ^*) to the problem (2.6).

Proof. Let $(m, \varpi) \in X$ satisfying the equation $e(m, \varpi) = 0$. According to the cost of tracking type in (2.1) we have

$$J(m, \varpi) \ge \frac{\delta}{2}\|\varpi\|_{L^2(Q_0)}^2.$$

From (3.1) we conclude that $\|m\|_{W(V)} \to \infty$ yields $\|\varpi\|_{L^2(Q_0)} \to \infty$. Then

$$J(m,\varpi)\to+\infty \text{ , as } \|(m,\varpi)\|_X\to\infty \quad (4.1)$$

The norm is weakly lower semi-continuous [41], so J is weakly lower semi-continuous. Since $J(m,\varpi)\geq 0$, for all $(m,\varpi)\in X$ holds, there exists $\kappa\geq 0$ defined by

$$\kappa=\inf\{J(m,\varpi)\mid(m,\varpi)\in X, e(m,\varpi)=0\} \text{ ,}$$

which implies the existence of a minimizing sequence $\{(m^n,\varpi^n)\}_{n\in N}$ in X such that

$$\kappa=\lim_{n\to\infty}J(m^n,\varpi^n) \text{ , } e(m^n,\varpi^n)=0 \text{ , } \forall n\in N.$$

Then it follows from (4.1) that there exists an element $(m^*,\varpi^*)\in X$ such that when $n\to\infty$,

$$m^n\to m^* \text{ in } W(V) \text{ weakly ,} \quad (4.2)$$

and

$$\varpi^n\to\varpi^* \text{ in } L^2(Q_0) \text{ weakly .} \quad (4.3)$$

(4.2) implies that

$$\int_0^T\left(m_t^n(t)-m_t^*(t),\phi\right)_{V^*,V}dt\to 0 \text{ , as } n\to\infty,\ \forall\phi\in L^2(V), \quad (4.4)$$

and

$$\int_0^T\left(m^n-m^*,\phi\right)_V dt\to 0, \text{ as } n\to\infty \text{ , } \forall\phi\in L^2(V). \quad (4.5)$$

Thanks to the facts that $W(V)$ is compactly embedded into $L^2(L^\infty)$ [39], we conclude that $m^n\to m^*$ strongly in $L^2(L^\infty)$. Since $m^n\to m^*$ weakly in $W(V)$, $\|m^n\|_{W(V)}$ is bounded [42]. As $W(V)$ is compactly embedded into $C(H)$ [38], we then derive $\|m^n\|_{C(H)}$ is bounded.

We have

$$\left|\int_0^T\int_\Omega\left(u^n u_x^n m^n-u^*u_x^*m^*\right)\phi dxdt\right|$$

$$=\frac{1}{2}\left|\int_0^T\int_\Omega\left(\left[(u^n)^2\right]_x m^n-\left[(u^*)^2\right]_x m^*\right)\phi dxdt\right|$$

$$\leq\frac{1}{2}\int_0^T\left|\int_\Omega\left[(u^n)^2\right]_x\left(m^n-m^*\right)\phi dx\right|dt$$

$$+\frac{1}{2}\int_0^T\left|\int_\Omega\left[(u^n)^2-(u^*)^2\right]_x m^*\phi dx\right|dt \quad (4.6)$$

$$\overset{\Delta}{=}I+II \text{ ,}$$

where $m^n=u^n-(u^n)_{xx}$ and $m^*=u^*-(u^*)_{xx}$.

First we observe that Young's inequality for convolutions yields

$$\begin{aligned}\|u^n-u^*\|_{L^\infty}&=\left\|\left(1-\partial_x^2\right)^{-1}\left(m^n-m^*\right)\right\|_{L^\infty}\\&=\left\|G*\left(m^n-m^*\right)\right\|_{L^\infty}\\&\leq\|G\|_{L^1}\|m^n-m^*\|_{L^\infty}\leq\|m^n-m^*\|_{L^\infty}\end{aligned} \quad (4.7)$$

and

$$\begin{aligned}\left\|(u^n-u^*)_x\right\|_{L^\infty}&=\left\|\left(\left(1-\partial_x^2\right)^{-1}\left(m^n-m^*\right)\right)_x\right\|_{L^\infty}\\&=\left\|\left(G*\left(m^n-m^*\right)\right)_x\right\|_{L^\infty}=\left\|G_x*\left(m^n-m^*\right)\right\|_{L^\infty}\\&\leq\|G_x\|_{L^1}\|m^n-m^*\|_{L^\infty}\leq\|m^n-m^*\|_{L^\infty},\end{aligned} \quad (4.8)$$

where $G=\dfrac{\cosh(x-\frac{1}{2})}{2\sinh(\frac{1}{2})}$ and $*$ denotes the convolution.

For the part I in (4.6), it follows from Hölder's inequality that

$$I = \frac{1}{2}\int_0^T \left| \int_\Omega \left[\left(u^n\right)^2 \right]_x \left(m^n - m^*\right) \phi dx \right| dt$$

$$\leq \int_0^T \left| \int_\Omega u^n u_x^n \left(m^n - m^*\right) \phi dx \right| dt$$

$$\leq \int_0^T \left\| u^n \left(m^n - m^*\right) \right\|_{L^\infty} \left| \int_\Omega u_x^n \phi dx \right| dt$$

$$\leq \int_0^T \left\| u^n \right\|_{L^\infty} \left\| m^n - m^* \right\|_{L^\infty} \left| \int_\Omega u^n \phi_x dx \right| dt \qquad (4.9)$$

$$\leq C\int_0^T \left\| u^n \right\|_V \left\| m^n - m^* \right\|_{L^\infty} \left\| u^n \right\|_H \left\| \phi_x \right\|_H dt$$

$$\leq C\int_0^T \left\| m^n \right\|_H^2 \left\| m^n - m^* \right\|_{L^\infty} \left\| \phi \right\|_V dt$$

$$\leq C\left\| m^n \right\|_{C(H)}^2 \left\| m^n - m^* \right\|_{L^2\left(L^\infty\right)} \left\| \phi \right\|_{L^2(V)}$$

$\to 0$, as $n \to \infty$, $\forall \phi \in L^2(V)$,

where we used the fact that $\left\| u^n \right\|_V \leq \left\| m^n \right\|_H \leq \left\| m^n \right\|_{C(H)}$ and $\left\| u^n \right\|_H \leq \left\| m^n \right\|_H \leq \left\| m^n \right\|_{C(H)}$ as in (3.10).

For the part II in (4.6) , we have

$$II = \frac{1}{2}\int_0^T \left| \int_\Omega \left[\left(u^n\right)^2 - \left(u^*\right)^2 \right]_x m^* \phi dx \right| dt$$

$$\leq C\int_0^T \left\| \left[\left(u^n\right)^2 - \left(u^*\right)^2 \right]_x \right\|_{L^\infty} \left\| m^* \right\|_H \left\| \phi \right\|_H$$

$$\leq C\left\| m^* \right\|_{C(H)} \int_0^T \left(\left\| u^n + u^* \right\|_{L^\infty} \left\| \left(u^n - u^*\right)_x \right\|_{L^\infty} \right.$$

$$\left. + \left\| \left(u^n + u^*\right)_x \right\|_{L^\infty} \left\| \left(u^n - u^*\right)_x \right\|_{L^\infty} \right) \left\| \phi \right\|_V dt$$

$$\leq C\left\| m^* \right\|_{C(H)} \int_0^T \left(\left\| u^n \right\|_V + \left\| u^* \right\|_V + \left\| u_x^n \right\|_{H^1} + \left\| u_x^* \right\|_{H^1} \right)$$

$$\times \left\| m^n - m^* \right\|_{L^\infty} \left\| \phi \right\|_V dt \qquad (4.10)$$

$$\leq C\left\| m^* \right\|_{C(H)} \int_0^T \left(\left\| m^n \right\|_H + \left\| m^* \right\|_H \right) \left\| m^n - m^* \right\|_{L^\infty} \left\| \phi \right\|_V dt$$

$$\leq C\left\| m^* \right\|_{C(H)} \left(\left\| m^n \right\|_{C(H)} + \left\| m^* \right\|_{C(H)} \right) \left\| m^n - m^* \right\|_{L^2\left(L^\infty\right)} \left\| \phi \right\|_{L^2(V)}$$

$\to 0$, as $n \to \infty$, $\forall \phi \in L^2(V)$,

where we have used (4.7) and (4.8).

Then it follows from (4.6) , (4.9) and (4.10) that

$$\int_0^T \int_\Omega \left(u^n u_x^n m^n - u^* u_x^* m^* \right) \phi sxdt$$

$\to 0$, as $n \to \infty$, $\forall \phi \in L^2(V)$ (4.11)

We also have

$$\left| \int_0^T \int_\Omega \left[\left(u^n\right)^2 m_x^n - \left(u^*\right)^2 m_x^* \right] \phi dxdt \right|$$

$$\leq \int_0^T \left| \int_\Omega \left(u^n\right)^2 \left(m_x^n - m_x^* \right) \phi dx \right| dt \qquad (4.12)$$

$$+ \int_0^T \left| \int_\Omega \left[\left(u^n\right)^2 - \left(u^*\right)^2 \right] m_x^* \phi dx \right| dt$$

$$\stackrel{\Delta}{=} III + IV .$$

For the part III and IV in (4.12) , we use Hölder's inequality to find

$$III = \int_0^T \left| \int_\Omega \left(u^n\right)^2 \left(m_x^n - m_x^* \right) \phi dx \right| dt$$

$$\leq \int_0^T \left| \int_\Omega \left(m^n - m^*\right) \left[2u^n u_x^n \phi + \left(u^n\right)^2 \phi_x \right] dx \right| dt$$

$$\leq 2\int_0^T \left\| m^n - m^* \right\|_{L^\infty} \left\| u^n \right\|_{L^\infty} \left| \int_\Omega u_x^n \phi dx \right| dt$$

$$+ \int_0^T \left\| m^n - m^* \right\|_{L^\infty} \left\| u^n \right\|_{L^\infty} \left| \int_\Omega u^n \phi_x dx \right| dt$$

$$\leq 2\int_0^T \left\| m^n - m^* \right\|_{L^\infty} \left\| u^n \right\|_V \left\| u^n \right\|_V \left\| \phi \right\|_H dt \quad (4.13)$$

$$+ \int_0^T \left\| m^n - m^* \right\|_{L^\infty} \left\| u^n \right\|_V \left\| u^n \right\|_H \left\| \phi \right\|_V dt$$

$$\leq C\int_0^T \left\| m^n - m^* \right\|_{L^\infty} \left\| m^n \right\|_H \left\| m^n \right\|_H \left\| \phi \right\|_V dt$$

$$\leq C\left\| m^n \right\|_{C(H)}^2 \left\| m^n - m^* \right\|_{L^2\left(L^\infty\right)} \left\| \phi \right\|_{L^2(V)}$$

$\to 0$, as $n \to \infty$, $\forall \phi \in L^2(V)$,

and

$$IV = \int_0^T \left| \int_\Omega \left[(u^n)^2 - (u^*)^2 \right] m_x^* \phi dx \right| dt$$

$$\le \int_0^T \left\| (u^n)^2 - (u^*)^2 \right\|_{L^\infty} \left| \int_\Omega m_x^* \phi dx \right| dt$$

$$\le \frac{1}{2} \int_0^T \left\| \left[(u^n)^2 - (u^*)^2 \right]_x \right\|_{L^\infty} \|m^*\|_H \|\phi\|_V dt$$

$$\le C \|m^*\|_{C(H)} \int_0^T \|u^n + u^*\|_{L^\infty} \|u^n - u^*\|_{L^\infty} \|\phi\|_V dt \quad (4.14)$$

$$\le C \|m^*\|_{C(H)} \int_0^T \left(\|u^n\|_V + \|u^*\|_V \right) \|m^n - m^*\|_{L^\infty} \|\phi\|_V dt$$

$$\le C \|m^*\|_{C(H)} \int_0^T \left(\|m^n\|_H + \|m^*\|_H \right) \|m^n - m^*\|_{L^\infty} \|\phi\|_V dt$$

$$\le C \|m^*\|_{C(H)} \left(\|m^n\|_{C(H)} + \|m^*\|_{C(H)} \right) \|m^n - m^*\|_{L^2(L^\infty)} \|\phi\|_{L^2(V)}$$

$\to 0$, as $n \to \infty$, $\forall \phi \in L^2(V)$.

Then we derive that

$$\int_0^T \int_\Omega \left[(u^n)^2 m_x^n - (u^*)^2 m_x^* \right] \phi dx dt$$
$$\to 0 \text{, as } n \to \infty \text{, } \forall \phi \in L^2(V) . \quad (4.15)$$

We infer from (4.3) that

$$\int_0^T \int_\Omega \left(B^* \varpi^n - B^* \varpi^* \right) \phi dx dt$$
$$\to 0 \text{, as } n \to \infty \text{, } \forall \phi \in L^2(V) \quad (4.16)$$

Thus from (4.4), (4.5), (4.11) and (4.15), we conclude that $e_1(m^*, \varpi^*) = 0$ in $L^2(V)$.

From $m^* \in W(V)$ we derive that $m^*(0) \in H$. Since $m^n \to m^*$ weakly in $W(V)$, we have $m^n(0) \to m^*(0)$ weakly in H as $n \to \infty$, that is

$$\left(m^n(0) \to m^*(0), \psi \right)_H \to 0 \text{, as } n \to \infty \text{, } \forall \psi \in H,$$

which implies that $e_2(m^*, \varpi^*) = 0$ in H. Thus we obtain that $e(m^*, \varpi^*) = 0$ in Y.

Therefore, we have shown that $(m^*, \varpi^*) \in Y$ is an optimal solution to the control problem (2.6). This completes the proof of the theorem.

5. CONCLUSION

In this work we investigate an optimal control problem for the viscous modified Novikov equation. The existence and uniqueness of a local regular solution to this equation is obtained and the existence of an optimal solution to the control problem is proved.

6. FUTURE RESEARCH WORK

This work is the base of the first-order necessary optimality condition, the second-order sufficient optimality condition and the numerical test. These issues would be worked on in the near future.

ACKNOWLEDGEMENTS

The authors would like to thank all the anonymous referees for their careful reading of the paper and constructive comments and suggestions which have helped to improve the representation of the paper. This work was supported by the Natural Science Foundation of China (No. 11171135, 10420130638).

COMPETING INTERESTS

The authors declare that there is no conflict of interests regarding the publication of this paper.

REFERENCES

1. Novikov V. Generalizations of the Camassa-Holm equation. J. Phys. A. 2009;42:342002.
2. Camassa R, Holm D. An integrable shallow water equation with peaked solitons. Phys. Rev. Lett. 1993;71:1661-1664.

3. Home ANW, Wang JP. Integrable peakon equations with cubic nonlinearity. J. Phys. A. 2008;41:372002.
4. Liu X, Liu Y, Qu C. Stability of peakons for the Novikov equation. J. Math. Pures Appl. 2014;101:172-187.
5. Hone W, Lundmark H, Szmigielski J. Explicit multipeakon solutions of Novikov cubically nonlinear integrable Camassa-Holm type equation. Dyn. Partial Differ. Equ. 2009;6:253-289.
6. Tiglay F. The periodic Cauchy problem for Novikov equation. Int. Math. Res. Not. IMRN. 2011;20:4633-4648.
7. Ni L, Zhou Y. Well-posedness and persistence properties for the Novikov equation. J. Differ. Equations. 2011; 250:3002-3021.
8. Wu X, Yin Z. Global weak solutions for the Novikov equation. J. Phys. A. 2011;44:055202.
9. Wu X, Yin Z. Well-posedness and global existence for the Novikov equation Ann. Sc. Norm. Super. Pisa Cl. Sci. 2012;3:707-727.
10. Himonas A, Holliman C. The Cauchy problem for the Novikov equation. Nonlinearity. 2012;25:449-479.
11. Yan W, Li Y, Zhang Y. Cauchy problem for the integrable Novikov equation. J. Differ. Equations. 2012;253:298-318.
12. Jiang Z, Ni L. Blow-up phenomenon for the integrable Novikov equation. J. Math. Anal. Appl. 2012;385:551-558.
13. Lai S, Li N, Wu Y. The existence of global strong and weak solutions for the Novikov equation. J. Math. Anal. Appl. 2013; 399:682-691.
14. Ott E, Sudan RN. Damping of solitary waves. Phys. Fluids. 1970;13:1432-1434.
15. Ghidaglia JM. Weakly damped forced Korteweg-de Vries equations behave as a finite dimensional dynamical system in the long time. J. Differ. Equations. 1988; 74:369-390.
16. Wu S, Yin Z. Global existence and blow-up phenomena for the weakly dissipative Camassa-Holm equation. J. Differ. Equations. 2009;246:4309-4321.
17. Wu S, Yin Z. Blow-up, blow-up rate and decay of the solution of the weakly dissipative Camassa-Holm equation. J. Math. Phys. 2006;47:1-12.
18. Wu S, Yin Z. Blow-up and decay of the solution of the weakly dissipative Degasperis-Procesi equation. SIAM J. Math. Anal. 2008;40:475-490.
19. Wu S, Escher J, Yin Z. Global existence and blow-up phenomena for a weakly dissipative Degasperis-Procesi equation. Discrete Cont. Dyn-B. 2009;12:633-645.
20. Wu S, Yin Z. Blow-up phenomena and decay for the periodic Degasperis-Procesi equation with weak dissipation. J. Nonlinear Math. Phys. 2008;15:28-49.
21. Yan W, Li Y, Zhang Y. Global existence and blow-up phenomena for the weakly dissipative Novikov equation. Nonlinear Anal. 2012;75:2464-2473.
22. Mi Y, Mu C. On the Cauchy problem for the modified Novikov equation with peakon solutions. J. Differ. Equations. 2013; 254:961-982.
23. Vedantham R. Optimal control of the viscous Burgers equation using an equivalent index method. J. Global Optim. 2000;18:255-263.
24. Volkwein S. Distributed control problems for the Burgers equation. Comput. Optim. Appl. 2001;18:115-140.
25. Hinze M, Volkwein S. Analysis of instantaneous control for the Burgers equation Nonlinear Anal. 2002;50:1-26.
26. Tian L, Shen C, Ding D. Optimal control of the viscous Camassa-Holm equation. Nonlinear Anal. RWA. 2009;10:519-530.
27. Tian L, Shen C. Optimal control of the viscous Degasperis-Procesi equation. J. Math. Phys. 2007;48:113513.
28. Shen C, Tian L, Gao A. Optimal control of the viscous Dullin-Gottwalld-Holm equation. Nonlinear Anal. RWA. 2010;11:480-491.
29. Ito K, Ravindran SS. A reduced-basis method for control problems governed by PDEs. Control and estimation of distributed parameter systems. Int. Ser. Numer. Math. 1998;126:153-168.
30. Atwell JA, King BB. Proper orthogonal decomposition for reduced basis feedback controller for parabolic equation. Math. Comput. Modelling. 2001;33:1-19.
31. Hu C, Temam R. Robust control of the Kuramoto-Sivashinsky equation. Dyn. Contin. Discrete Impuls. Syst. B: Appl. Algoritheorems. 2001;8:315-338.
32. Ghattas O, Bark JH. Optimal control of two- and three-dimensional incompressible Navier-Stokes flows. J. Comput. Phys. 1997;136:231-244.
33. Ryu SU, Yagi A. Optimal control of Keller-Segel equations. J. Math. Anal. Appl. 2001;256:45-66.

34. Lagnese JE, Leugering G. Time-domain decomposition of optimal control problems for the wave equation. Systems Control Lett. 2003;48:229-242.
35. Oksendal B. Optimal control of stochastic partial differential equations. Stoch. Anal. Appl. 2005;23:165-179.
36. Zhao X, Liu C. Optimal control problem for viscous Cahn-Hilliard equation. Nonlinear Anal. 2011;74:6348-6357.
37. Temam R. Infinite-dimensional dynamical systems in mechanics and physics. Springer-Verlag: New York; 1988.
38. Dautray R, Lions JL. Mathematical analysis and numerical methods for science and technology. Volume 5: Evolution Problems I. Springer-Verlag: Berlin; 1992.
39. Temam R. Navier-Stokes equations, in: Studies in Mathematics and its Applications, North-Holland, Amsterdam; 1979.
40. Lions JL. Optimal control of systems governed by partial differential equations. Springer-Verlag: Berlin; 1971.
41. Wouk A. A course of applied functional analysis. Wiley-Interscience: New York; 1979.
42. Kreyszig E. Introductory functional analysis with applications. John Wiley & Sons; 1978.

Peer-review history:
The peer review history for this paper can be accessed here:
http://www.sciencedomain.org/review-history.php?iid=904&id=41&aid=7605