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A Note on Multiplication Operators between Lorentz-Karamata Spaces

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Abstract

Multiplication operators between some spaces isomorphic to Lorentz-Karamata spaces are discussed. We also describe the multiplication operators on Lorentz-Karamata spaces which are compact and Fredholm.

Keywords: Lorentz-Karamata space; multiplication operators; compact operator; distribution function; slowly varying function.

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1 Introduction

Let $F(X)$ be a vector space of all complex-valued functions on a σ-finite measure space $(X, \mathcal{A}, \mu)$. Let $u : X \to C$ be a measurable function on X such that $u \cdot f \in F(X)$ whenever $f \in F(X)$. This gives rise to a linear transformation $M_u : F(X) \to F(X)$ defined as $M_u(f) = u \cdot f$, where the product of functions is pointwise. In case $F(X)$ is a topological vector space and M_u is continuous, we call it a multiplication operator induced by u. The study of multiplication operators is interesting as well as demanding with its close association with various classes of operators particularly Toeplitz operators, Hankel operators, slant Toeplitz operators, slant Hankel operators, composition operators

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and the fact that every normal operator is similar to multiplication operator [1,2,3,4]. With the existence and introduction of various function spaces [5], it is always interesting to extend the study of multiplication operator on them. Motivated by the approach and direction of research by the mathematician in [6,7,8,9,10,11,12,13,14], an effort has been made in the paper to discuss the behavior of this class of operators on Lorentz-Karamata spaces.

Let f be a complex-valued measurable function defined on a σ-finite measure space $(X, \mathcal{A}, \mu)$. The *distribution function* μ_f *of* f *is given by*

$$\mu_f(s) \;=\; \mu\{x \in X : |f(x)| > s\}$$

for $s \geq 0$. By f^* we mean the *non-increasing rearrangement* of f given as

$$f^*(t) \;=\; \inf\{s > 0 : \mu_f(s) \leq t\},\ t \geq 0.$$

Let $L(p,q),\ 1 < p \leq \infty,\ 1 \leq q \leq \infty$, is the set of all complex-valued measurable functions f on X such that $\|f\|^*_{pq} < \infty$, where

$$\|f\|^*_{pq} = \begin{cases} \left\{ \frac{q}{p} \int_0^\infty (\ t^{1/p} f^*(t))^q\ \frac{dt}{t} \right\}^{1/q}, & 1 < p < \infty,\ \ 1 \leq q < \infty, \\ \sup\limits_{t>0} t^{1/p} f^*(t), & 1 < p \leq \infty,\ \ q = \infty. \end{cases}$$

$L_{p,q}$ spaces are linear spaces and $\|\cdot\|^*_{pq}$ is a quasi-norm and is a norm for $1 \leq q \leq p < \infty$ *or* $p = q = \infty$.

For $t > 0$, let

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds \quad and \quad f^{**}(0) = f^*(0).$$

For $1 < p \leq \infty,\ 1 \leq q \leq \infty$, and for measurable function f on X define $\|f\|_{pq}$ as

$$\|f\|_{pq} \;=\; \begin{cases} \left\{ \frac{q}{p} \int_0^\infty \left(t^{1/p} f^{**}(t)\right)^q\ \frac{dt}{t} \right\}^{1/q}, & 1 < p < \infty,\ 1 \leq q < \infty \\ \sup_{t>0} t^{1/p} f^{**}(t), & 1 < p \leq \infty,\ q = \infty \end{cases}$$

Lorentz space denoted by $L_{(p,q)}(X, \mathcal{A}, \mu)$ (or shortly $L_{(p,q)}$) is defined to be the vector space of all (equivalence classes of) measurable functions f on X such that $\|f\|_{pq} < \infty$. Also $\|\cdot\|_{pq}$ is a norm and $L(p,q)$ is a Banach space with respect to this norm. The L^p- spaces for $1 < p \leq \infty$ are equivalent to the spaces $L_{(p,p)}$. For more on Lorentz spaces one can refer to [15,16,17,18].

Definition 1.1. A positive and Lebesgue measurable function b is said to be slowly varying (s.v.) on (0,∞) in the sense of Karamata if, for each $\epsilon > 0, t^\epsilon b(t)$ is equivalent to a non-decreasing function and $t^{-\epsilon} b(t)$ is equivalent to a non-increasing function on (0,∞).

The detailed study of Karamata theory, properties and examples of slowly varying functions can be found in [19,20,21,22,23]. Given a s.v. function b on (0,∞), we denote by $\gamma_b(t)$, the positive function defined by

$$\gamma_b(t) = b(\max\{t, 1/t\})$$

for all $t > 0$. It is known that any slowly varying function b on $(0, \infty)$ is equivalent to a slowly varying continuous function $\tilde{b}$ on $(0, \infty)$. Consequently, without loss of generality, we assume that all slowly varying functions are continuous functions in $(0, \infty)$ [22]. We shall need the following property of s.v. functions, for which we refer to [21, Lemma 3.1].

Lemma 1.2. *Let b be a slowly varying function on* $(0, \infty)$.

1. *: Let* $r \in R$. *Then* b^r *is a slowly varying function on* $(0, \infty)$ *and* $\gamma_b^r(t) = \gamma_{b^r}(t)$ *for all* $t > 0$.

2. *: Given positive numbers* ϵ *and* k, $\gamma_b(kt) \approx \gamma_b(t)$ *i.e., there are positive constants* c_ϵ *and* C_ϵ *such that*

$$c_\epsilon \min\{k^{-\epsilon}, k^{\epsilon}\}\gamma_b(t) \leq \gamma_b(kt) \leq C_\epsilon \max\{k^{-\epsilon}, k^{\epsilon}\}\gamma_b(t) \tag{1.1}$$

for all $t > 0$.

3. *: Let* $\alpha > 0$. *Then*

$$\int_0^t \tau^{\alpha-1}\gamma_b(\tau)d\tau \approx t^{\alpha}\gamma_b(t) \ \text{and} \int_t^\infty \tau^{-\alpha-1}\gamma_b(\tau)d\tau \approx t^{-\alpha}\gamma_b(t) \tag{1.2}$$

for all $t > 0$.

Let b be a slowly varying function on $(0, \infty)$. Let $L_{p,q;b}$, $1 < p \leq \infty,\ 1 \leq q \leq \infty$, be the set of all complex-valued measurable functions f on X such that $\|f\|_{p,q;b} < \infty$, where

$$\|f\|_{p,q;b} = \begin{cases} \left\{ \int_0^\infty (\ t^{1/p}\gamma_b(t)f^*(t))^q \ \frac{dt}{t} \right\}^{1/q}, & 1 < p < \infty,\ 1 \leq q < \infty, \\ \sup\limits_{t>0} t^{1/p}\gamma_b(t)f^*(t), & 1 < p \leq \infty,\ q = \infty. \end{cases}$$

$L_{p,q;b}$ spaces are linear spaces and $\| \cdot \|_{p,q;b}$ is a quasi-norm. The Lorentz-Karamata (LK) space $L_{(p,q;b)}, (X, \mathcal{A}, \mu)$ (or shortly $L_{(p,q;b)}$) is defined to be the set of complex-valued measurable functions f on X for which

$$\|f\|_{(p,q;b)} = \begin{cases} \left\{ \int_0^\infty (\ t^{1/p}\gamma_b(t)f^{**}(t))^q \ \frac{dt}{t} \right\}^{1/q}, & 1 < p < \infty,\ 1 \leq q \leq \infty, \\ \sup\limits_{t>0} t^{1/p}\gamma_b(t)f^{**}(t), & 1 < p \leq \infty,\ q = \infty. \end{cases}$$

is finite.

The Lorentz, Lorentz-Zygmund and generalised Lorentz-Zygmund spaces are all special cases of these spaces, obtained by making particular choices of the slowly varying function b. With this tendency of LK spaces to coincide with various spaces, the study over these spaces becomes more demanding. It is also known that the LK spaces endowed with a convenient norm ($\| \cdot \|^*_{(p,q;b)}$), are rearrangment-invariant Banach function spaces with upper and lower Boyd indices both equal to $\frac{1}{p}$ and have absolutely continuous norm when $p \in (1, \infty)$ and $q \in [1, \infty)$. Also, the dual $(L_{(p,q;b)})^*$ coincides with the associate space $L_{(p',q';b^{-1})}$, where $1 < p, q < \infty$ and p', q' are the conjugate exponent of p, q respectively. It is clear that, for $0 < p < \infty$, the LK space $L_{(p,q;b)}$ contains the characteristic function of every measurable subset of X with finite measure and hence, by linearity, every μ-simple function. In this case, with a little thought, it is easy to see that the set of simple functions is dense in the LK space as the LK spaces have absolutely continuous norm for $p \in (1, \infty)$ and $q \in [6, \infty)$.

Multiplication operators have been studied on various function spaces in [7,11,13,14,24], and references therein. Along the line of their arguments, we study the multiplication operators on the Lorentz-Karamata spaces $L_{p,q;b}, 1 < p \leq \infty, 1 \leq q \leq \infty$. In the last section, which is the main section of the paper, we discuss some spaces isomorphic to the spaces $L_{p,q;b}$ and then characterize the boundedness of multiplication operators between them in terms of the inducing function. The symbol $L_\infty(\mu)$ is used to denote the space of all essentially bounded complex-valued measurable functions on the measure space X.

2 Multiplication Operators

In this section, we study some properties of the multiplication operators M_u in terms of the inducing function u. In [16], the study is made with respect to the positive function generated by $\gamma_b(t) = 1 + |\log t|$, which contains the results and ideas generated in [11]. However, we find that results can be extended in more general setting. The results obtained in this section follow almost along the lines of arguments applied for the same results in case of Lorentz spaces [7], Orlicz spaces [14] and Orlicz-Lorentz spaces [8], Lorentz-Zygmunds spaces [16].

Theorem 2.1. *The linear transformation $M_u : f \to u.f$ on the Lorentz-Karamata space $L_{(p,q;b)}$, $1 < p \leq \infty, 1 \leq q \leq \infty$ is bounded if and only if u is essentially bounded. Moreover, $\|M_u\| = \|u\|_\infty$.*

Proof. Suppose that u is essentially bounded. Now for each $f \in L_{(p,q;b)}$, $1 < p \leq \infty, 1 \leq q \leq \infty$, we have $(u.f)^{**}(t) \leq \|u\|_\infty f^{**}(t)$ for each $t > 0$. This provides $\|M_u f\|_{(p,q;b)} \leq \|u\|_\infty \|f\|_{(p,q;b)}$ so that M_u is bounded. Conversely, suppose that M_u is a bounded operator on $L_{p,q;b}, 1 < p \leq \infty, 1 \leq q \leq \infty$. If u is not essentially bounded, then for each natural number n, the set $E_n = \{x \in X : |u(x)| \geq n\}$ has positive measure.Thus

$$\begin{aligned} \chi^{**}_{E_n}(t) &= \frac{1}{t}\int_0^t \chi^*_{E_n}(s)ds \\ &= \begin{cases} 1, & \text{if } 0 \leq t < \mu(E_n) \\ \frac{1}{t}\,\mu(E_n), & \text{if } t \geq \mu(E_n). \end{cases} \end{aligned}$$

Also, $\|\chi_{E_n}\|^q_{(p,q;b)} = \int_0^{\mu(E_n)} t^{\frac{q}{p}-1}(\gamma_b(t))^q dt + (\mu(E_n))^q \int_{\mu(E_n)}^\infty t^{-q(1-\frac{1}{p})-1}(\gamma_b(t))^q dt \approx (\mu(E_n))^{\frac{q}{p}}(\gamma_b(\mu(E_n)))^q + (\mu(E_n))^q(\mu(E_n))^{-q(1-\frac{1}{p})}(\gamma_b(\mu(E_n)))^q = 2(\mu(E_n))^{\frac{q}{p}}(\gamma_b(\mu(E_n)))^q < \infty$. Further, for $t > 0$, $(u\chi_{E_n})^*(t) \geq n\chi^*_{E_n}(t)$ and this gives that $\|M_u\chi_{E_n}\|^q_{(p,q;b)} \geq n^q\|\chi_{E_n}\|^q_{(p,q;b)}$. This contradicts the boundedness of M_u. Hence the result.

Moreover, along the lines of proof in the case of Lorentz space [7], one can show that $\|M_u\| = \|u\|_\infty$. □

In order to characterize the invertibility of M_u in terms of u, we need the following result, proof of which follows along the lines of arguments applied in case of Lorentz-Zygmunds spaces in [6] motivated by the arguments of Conway [25, Proposition 12.4, p. 57] .

Theorem 2.2. *The set of all multiplication operators on the Lorentz-Karamata space $L_{(p,q;b)}, 1 < p < \infty, 1 \leq q < \infty$, is a maximal abelian subalgebra of $\mathfrak{B}(L_{(p,q;b)})$, the Banach algebra of all bounded linear operators on $L_{(p,q;b)}$.*

As a consequence of Theorem 2.2, we have the following.

Corollary 2.3. *The multiplication operator M_u on $L_{(p,q;b)}, 1 < p < \infty, 1 \leq q < \infty$ is invertible if and only if u is invertible in $L_\infty(\mu)$.*

We find that there is a dearth of compact multiplication operators on Lorentz-Karamata space once the measure is non-atomic. By adopting the technique used in [7], we can easily obtain the following results.

Theorem 2.4. *Multiplication operator M_u on Lorentz-Karamata space $L_{(p,q;b)}, 1 < p \leq \infty, 1 \leq q \leq \infty$ is compact if and only if $L_{(p,q;b)}(u, \epsilon)$, is finite dimensional for each $\epsilon > 0$, where*

$$(u, \epsilon) = \{x \in X : |u(x)| \geq \epsilon\} \text{ and } L_{(p,q;b)}(u, \epsilon) = \{f\chi_{(u,\epsilon)} : f \in L_{(p,q;b)}\}.$$

Corollary 2.5. *If for each $\epsilon > 0$, the set (u, ϵ) contains only finitely many atoms then M_u is a compact multiplication operator on the Lorentz-Karamata space $L_{(p,q;b)}$.*

Corollary 2.6. *If μ is a non-atomic measure, then the only compact multiplication operator on the Lorentz-Karamata space $L_{(p,q;b)}$ is the zero operator.*

Theorem 2.7. *Multiplication operator M_u on Lorentz-Karamata space $L_{(p,q;b)}, 1 < p \leq \infty, 1 \leq q \leq \infty$ has closed range if and only if there exists a $\delta > 0$ such that $|u(x)| \geq \delta$ on $S = \{x \in X : u(x) \neq 0\}$, the support of u.*

We use Theorem 2.7 to obtain the following.

Theorem 2.8. *Suppose that μ is a non-atomic measure. If M_u is a multiplication operator on the Lorentz-Karamata space $L_{(p,q;b)}, 1 < p \leq \infty, 1 \leq q \leq \infty$ with $R(M_u)$ closed and codim$(R(M_u)) < \infty$ then $|u(x)| \geq \delta$ a.e. on X for some $\delta > 0$, where $R(M_u)$ denotes the range of M_u and codim$(R(M_u))$ is the co-dimension of M_u.*

Proof. Suppose that $R(M_u)$ is closed and codim$(R(M_u))< \infty$. Then there exists $\delta > 0$ such that $|u(x)| \geq \delta$ a.e. on the support S of u. Hence, it is enough to show that $\mu(S^c) = 0$, where S^c is the set defined as $S^c = \{x \in X : u(x) = 0\}$. First, we claim that M_u is onto. If possible M_u be not onto and let $f_\circ \in L_{(p,q;b)} \setminus R(M_u)$. Since, $R(M_u)$ is closed, we can find a function $g_\circ \in L_{(\acute{p},\acute{q},b^{-1})}$, the conjugate space, where $\frac{1}{p} + \frac{1}{\acute{p}} = \frac{1}{q} + \frac{1}{\acute{q}} = 1$ such that $\int f_\circ g_\circ d\mu = 1$ and $\int (M_u f) g_\circ d\mu = 0$ for all $f \in L_{(p,q;b)}$. This gives that the set $E_\epsilon = \{x \in X : Re(f_\circ g_\circ)(x) \geq \epsilon\}$ has positive measure for some $\epsilon > 0$. Choose a sequence $\{E_n\}_n$ of subsets (which is feasible being μ non-atomic) of E_ϵ with $0 < \mu(E_n) < \infty$ and $E_m \cap E_n = \phi (m \neq n)$. Let $g_n = \chi_{E_n} g_\circ$. Since

$$Re \int f_\circ g_n d\mu = Re \int_{E_n} f_\circ g_\circ d\mu \geq \epsilon \mu(E_n) > 0,$$

hence each $g_n \in L_{(\acute{p},\acute{q},b^{-1})}$ is nonzero. Furthermore, for each $f \in L_{(p,q;b)}$, $\chi_{E_n} f \in L_{(p,q;b)}$ and so

$$(M_u^* g_n)(f) = g_n(M_u f) = \int (M_u f) g_n d\mu = \int (M_u f \chi_{E_n}) g_\circ d\mu = 0,$$

where M_u^* is the conjugate operator of M_u. This implies $g_n \in N(M_u^*)$, the null space of M_u^*. Thus, the sequence $\{g_n\}_n$ forms a linearly independent subset of $N(M_u^*)$. This contradicts the fact that $\dim N(M_u^*) = codim R(M_u) < \infty$. Hence M_u is onto.

Now, it is easily seen that $\mu(S^c) = 0$. For, if $\mu(S^c) > 0$, then there exists a subset A of S^c with $0 < \mu(A) < \infty$. Then $\chi_A \in L_{(p,q;b)} \setminus R(M_u)$, which contradicts the fact that M_u is onto. Therefore $\mu(S^c) = 0$, and so we have $|u(x)| \geq \delta$ a.e. on X. ☐

The following assertion can be easily drawn from here.

Corollary 2.9. *Let $M_u, u \in L_\infty(\mu)$ be a multiplication operator on the Lorentz space $L_{(p,q;b)}$, $1 < p < \infty$, $1 < q < \infty$, where μ is a non-atomic measure. Then the following conditions are equivalent:*

1. *M_u is an invertible operator.*
2. *M_u is a Fredholm operator.*
3. *$R(M_u)$ is closed and $codim R(M_u) < \infty$.*
4. *$|u(x)| \geq \delta$ a.e. on X for some $\delta > 0$.*

3 Multiplication Operators on Some Special Spaces

The goal of present section is to discuss multiplication operators between Lorentz-Karamata spaces. Theorem 3.4 provides a necessary condition on the inducing function to induce a multiplication operator between $L_{p,1;b}$ and $L_{s,1;b}$ for $s \geq p > 1$. We discuss some spaces isomorphic to $L_{p,q;b}$ for some specific values of p and q and the techniques used here are not new and motivated by the work of Stein and Weiss in [3] to extend the Marcinkiewicz interpolation theorem.

Consider $M^p_{r,b}$, $1 \leq p \leq r < \infty$, the set of real-valued functions defined on $X = [0, 2\pi]$ such that $\|f\|_{M^p_{r,b}} < \infty$, where

$$\|f\|_{M^p_{r,b}} = sup_{x>0} \left\{ \frac{r}{px^{\frac{1}{p}}} \int_0^x \left(\gamma_b(t) f^*(t) t^{\frac{1}{p}}\right)^r \frac{dt}{t} \right\}^{\frac{1}{r}}$$

We observe the following.

Lemma 3.1. *$\|\cdot\|_{M^p_{r,b}}$ is a quasi-norm on $M^p_{r,b}$.*

Proof. It is obvious from the definition of non-increasing rearrangement and slowly varying function that for each f, $\|f\|_{M^p_{r,b}} \geq 0$. Moreover, $\|f\|_{M^p_{r,b}} = 0$ implies that for all $0 < x \leq 2\pi$, $\int_0^x (\gamma_b(t) f^*(t) t^{\frac{1}{p}})^r \frac{dt}{t} = 0$. Being $\gamma_b(t) t^{\frac{1}{p}}$ a positive function, we have $f^* = 0$. As a consequence $f = 0$. Also, for each nonzero real number k and $f \in M^p_{r,b}$, $(kf)^* = |k| f^*$. Thus the homogeneity condition $\|kf\|_{M^p_{r,b}} = |k| \|f\|_{M^p_{r,b}}$ follows trivially. Let $f, g \in M^p_{r,b}$. Applying the fact that $(f+g)^* \leq f^*(t/2) + g^*(t/2)$, one can see that for any $x \in (0, 2\pi]$,

$$\begin{aligned}
\int_0^x \left(\gamma_b(t)(f+g)^*(t) t^{1/p}\right)^r \frac{dt}{t} &\leq 2^{r-1} \left(\int_0^x \left(\gamma_b(t) f^*(t/2) t^{1/p}\right)^r \frac{dt}{t} + \int_0^x \left(\gamma_b(t) g^*(t/2) t^{1/p}\right)^r \frac{dt}{t} \right) \\
&\leq 2^{r/p+r-1} \left(\int_0^{\frac{x}{2}} \left(\gamma_b(2t) f^*(t) t^{1/p}\right)^r \frac{dt}{t} + \int_0^{\frac{x}{2}} \left(\gamma_b(2t) g^*(t) t^{1/p}\right)^r \frac{dt}{t} \right) \\
&\approx 2^{r/p+r-1} \left(\int_0^x \left(\gamma_b(t) f^*(t) t^{1/p}\right)^r \frac{dt}{t} + \int_0^x \left(\gamma_b(t) g^*(t) t^{1/p}\right)^r \frac{dt}{t} \right).
\end{aligned}$$

Now the fact $(a+b)^{1/r} \leq a^{1/r} + b^{1/r}$ for $a, b > 0$ yields that

$$\|f+g\|_{M^p_{r,b}} \leq 2^{\frac{r}{p}+r-1} (\|f\|_{M^p_{r,b}} + \|g\|_{M^p_{r,b}})$$

with $2^{\frac{r}{p}+r-1} > 1$. This completes the result. □

It is interesting to observe the following.

Theorem 3.2. *$M^p_{r,b} \cong L_{pr',\infty;b}$, where $r, r' \geq 1$, $\frac{1}{r} + \frac{1}{r'} = 1$*

Proof. Suppose $g \in M^p_{r,b}$ so that $C = \|g\|_{M^p_{r,b}} < \infty$. Now for each $x > 0$.

$$\begin{aligned}
C &\geq \left(\frac{r}{px^{\frac{1}{p}}} \int_0^x \left(\gamma_b(t) g^*(t) t^{1/p}\right)^r \frac{dt}{t}\right)^{\frac{1}{r}} \\
&\geq \left(\frac{r}{px^{\frac{1}{p}}} (g^*(x))^r \int_0^x \left(\gamma_b(t) t^{1/p}\right)^r \frac{dt}{t} \right)^{\frac{1}{r}} \\
&\simeq \left(\frac{r}{p} (g^*(x))^r (\gamma_b(x))^r x^{\frac{r}{p} - \frac{1}{p}} \right)^{\frac{1}{r}} = (\frac{r}{p})^{\frac{1}{r}} g^*(x) \gamma_b(x) x^{\frac{1}{pr'}}.
\end{aligned}$$

Thus $(\frac{r}{p})^{\frac{1}{r}} g^*(x)\gamma_b(x)x^{\frac{1}{pr'}} \leq C$ for all $x > 0$. This provides that $\|g\|_{pr',\infty;b} \leq C(\frac{p}{r})^{\frac{1}{r}}$ so that $g \in L_{pr',\infty;b}$.

Conversely, if $g \in L_{(pr',\infty;b)}$ then the constant $C^* = \|g\|_{pr',\infty;b}(\frac{p}{r})^{\frac{1}{r}} > 0$ satisfies $g^*(t)\gamma_b(t)t^{1/pr'} \leq C^*$ for all $t > 0$. Thus $t^{1/p}g^*(t)\gamma_b(t) \leq C^* t^{\frac{1}{p}-\frac{1}{pr'}} = C^* t^{1/rp}$. This means that $(g^*(t)\gamma_b(t)t^{1/p})^r \leq C^{*r}t^{1/p}$ for all $t > 0$. Now, for all $x > 0$

$$\left(\frac{r}{px^{\frac{1}{p}}}\int_0^x (\gamma_b(t)g^*(t)t^{\frac{1}{p}})^r \frac{dt}{t}\right)^{\frac{1}{r}} \leq \left\{\frac{r}{px^{\frac{1}{p}}}\int_0^x (C^{*r}t^{\frac{1}{p}})^r \frac{dt}{t}\right\}^{\frac{1}{r}} = C^* r^{\frac{1}{r}}.$$

As a consequence of this $\|g\|_{M^p_{r,b}} < C^* r^{\frac{1}{r}} < \infty$ and $g \in M^p_{r,b}$. Hence the result. □

It is apparent to conclude the following from Theorem 3.2.

Corollary 3.3. *$M^p_{\infty,b} \cong L_{p,\infty;b}$ and $M^p_{1,b} \cong L_{\infty;b}(= L_{\infty,\infty;b})$.*

With all this preparation, we come to the main task of the section. For a measurable real-valued function u defined on $[0, 2\pi]$, $(u\chi_{[0,x]})^*(t) \leq u^*(t)$ for each $t > 0$ and $(u\chi_{[0,x]})^*(t) = 0$ for each $t > x$, where $x \in [0, 2\pi]$. This implies that

$$\begin{aligned}\int_0^{2\pi} \gamma_b(t)(M_u\chi_{[0,x]})^*(t)t^{\frac{1}{s}-1}dt &= \int_0^x \gamma_b(t)(M_u\chi_{[0,x]})^*(t)t^{\frac{1}{s}-1}dt \\ &\leq \int_0^x \gamma_b(t)u^*(t)t^{\frac{1}{s}-1}dt.\end{aligned}$$

We say that a measurable real-valued function u defined on $[0, 2\pi]$ is having the property (P) if for each non-negative function h defined on $[0, 2\pi]$,

$$\int_0^x h(t)(u\chi_{[0,x]})^*(t)dt = \int_0^x h(t)u^*(t)dt,$$

where $x \in [0, 2\pi]$. It is apparent to see that in case u satisfies $(u\chi_{[0,x]})^*(t) \geq u^*(t)$ for each $t < x$ then it has the property (P). There are functions satisfying this property, in particular, every non-increasing function u satisfies $(u\chi_{[0,x]})^*(t) \geq u^*(t)$ for each $t < x$ and hence has the property (P). Now, we have the following.

Theorem 3.4. *Let $s \geq p > 1$ and u be a non-increasing real-valued function u having the property (P). Then a necessary condition for the multiplication operator $M_u : L_{p,1;b} \to L_{s,1;b}$ to be bounded is that $u \in L_{\infty,b}$.*

Proof. It is convenient to use $M^p_{1,b}$ in place of $L_{\infty,b}$. Assume that $\|M_u f\|_{s,1;b} \leq C\|f\|_{p,1;b}$ for each $f \in L_{p,1;b}$ and some $C > 0$. Let $x \in (0, 2\pi]$ and $f = \chi_{[0,x]}$. Now for each $t > 0$,

$$\begin{aligned}\int_0^x \gamma_b(t)u^*(t)t^{\frac{1}{s}-1}dt &= \int_0^x \gamma_b(t)(M_u\chi_{[0,x]})^*(t)t^{\frac{1}{s}-1}dt \\ &\leq \|M_u f\|_{s,1;b} \leq C\|f\|_{p,1;b} \\ &\leq C\int_0^x \gamma_b(t)\chi^*_{[0,x]}(t)t^{\frac{1}{p}-1}dt \\ &\approx C(x)^{\frac{1}{p}}\gamma_b(x) \approx C(2\pi)^{\frac{1}{p}}\gamma_b(2\pi).\end{aligned}$$

This yields that

$$\int_0^x \gamma_b(t)u^*(t)t^{\frac{1}{p}-1}t^{\frac{p-s}{ps}}dt = \int_0^x \gamma_b(t)u^*(t)t^{\frac{1}{s}-1}dt \leq C(x)^{\frac{1}{p}}\gamma_b(x).$$

Since $t \to t^{\frac{p-s}{ps}}$ is decreasing on $[0,x]$, $0 < x \leq 2\pi$, so we have $x^{\frac{p-s}{ps}} \int_0^x \gamma_b(t)u^*(t)t^{\frac{1}{p}-1}dt \leq \int_0^x \gamma_b(t)u^*(t)t^{\frac{1}{p}-1}t^{\frac{p-s}{ps}}\,dt$. This along with the definition of γ_b and the facts that for each $\epsilon > 0, t^\epsilon b(t)$ is equivalent to a non-decreasing function and $t^{-\epsilon}b(t)$ is equivalent to a non-increasing function on $(0,\infty)$, provides that

$$\begin{aligned}\frac{1}{px^{\frac{1}{p}}}\int_0^x \gamma_b(t)u^*(t)t^{\frac{1}{p}-1}dt &\leq \frac{C}{p}x^{\frac{s-p}{ps}}\gamma_b(x)\\ &\leq \begin{cases}\frac{C}{p}(1/x)^{-\epsilon}b(1/x), & x<1,\\ \frac{C}{p}x^\epsilon b(x), & x\geq 1\end{cases}\\ &\leq \frac{C}{p}K,\end{aligned}$$

where $\epsilon = \frac{s-p}{ps}$ and K is a constant given by $K = \max\{\gamma_b(1), (2\pi)^\epsilon\gamma_b(2\pi)\}$. This on taking supremum over all $x > 0$ yields that $u \in M^p_{1,b}$. □

It is interesting to see that if $s \geq p > 1$ then for each non-increasing real-valued function u, $\|M_u\chi_{[0,x]}\|_{M^s_{1,b}} \leq \|\chi_{[0,x]}\|_{M^s_{1,b}}$, where $0 < x \leq 2\pi$. This, we can see as if $w > 0$ then

$$\frac{1}{sw^{1/s}}\int_0^w \gamma_b(t)(u\chi_{[0,x]})^*(t)t^{\frac{1}{s}-1}dt \leq \frac{1}{pw^{\frac{1}{s}}}\int_0^w \gamma_b(t)(u\chi_{[0,x]})^*(t)t^{\frac{1}{s}-1}dt.$$

Now in case $w < x$, then

$$\begin{aligned}\frac{1}{pw^{\frac{1}{s}}}\int_0^w \gamma_b(t)(u\chi_{[0,x]})^*(t)t^{\frac{1}{s}-1}dt &\leq \frac{1}{pw^{\frac{1}{s}}}\int_0^w \gamma_b(t)\chi^*_{[0,x]}(t)t^{\frac{1}{s}-1}dt\\ &= \frac{1}{pw^{\frac{1}{s}}}\int_0^w \gamma_b(t)t^{\frac{1}{s}-1}dt \;=\; \frac{1}{p}\gamma_b(w)\\ &= \frac{1}{pw^{\frac{1}{p}}}\int_0^w \gamma_b(t)\chi^*_{[0,x]}(t)t^{\frac{1}{p}-1}dt\\ &\leq \|\chi_{[0,x]}\|_{M^s_{1,b}}.\end{aligned}$$

Also, we find that in case $w \geq x$, we have

$$\begin{aligned}\frac{1}{pw^{\frac{1}{s}}}\int_0^w \gamma_b(t)(u\chi_{[0,x]})^*(t)t^{\frac{1}{s}-1}dt &= \frac{1}{pw^{\frac{1}{s}}}\int_0^x \gamma_b(t)(u\chi_{[0,x]})^*(t)t^{\frac{1}{s}-1}dt\\ &\leq \frac{1}{pw^{\frac{1}{s}}}\int_0^x \gamma_b(t)\chi^*_{[0,x]}(t)t^{\frac{1}{s}-1}dt\\ &= (\frac{x}{w})^{\frac{1}{s}}\frac{1}{p}\gamma_b(x) \leq \frac{1}{p}\gamma_b(x)\\ &= \frac{1}{px^{1/p}}\int_0^x \gamma_b(t)\chi^*_{[0,x]}(t)t^{\frac{1}{p}-1}dt\\ &\leq \|\chi_{[0,x]}\|_{M^s_{1,b}}.\end{aligned}$$

As a consequence, we have $\sup_{w>0}\{\frac{1}{sw^{1/s}}\int_0^w \gamma_b(t)(u\chi_{[0,x]})^*(t)t^{\frac{1}{s}-1}dt\} \leq \|\chi_{[0,x]}\|_{M^s_{1,b}}$, equivalently, $\|M_u\chi_{[0,x]}\|_{M^s_{1,b}} \leq \|\chi_{[0,x]}\|_{M^s_{1,b}}$.

4 Conclusion

Multiplication operators are discussed on more general Banach function spaces, the Loretnz-Karamata spaces, which has Lorentz, Lorentz-Zygmund and Generalized Lorentz-Zygmund spaces as its particular cases. Along with the routine behaviour of multiplication operators on Lorentz-Karamata spaces, we obtained some spaces isomorphic to these spaces that helped us to characterize the boundedness of multiplication operators between Lorentz-Karamata spaces.

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Competing Interests

The authors declare that no competing interests exist.

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