British Journal of Mathematics & Computer Science
17(4): 1-6, 2016, Article no.BJMCS.26697
ISSN: 2231-0851

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Cofinitely G-supplemented Modules

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/BJMCS/2016/26697
Editor(s):
(1) Dijana Mosic, Department of Mathematics, University of Nis, Serbia.
Reviewers:
(1) Ergul Turkmen, Amasya University, Turkey.
(2) Burcu Nisanci Turkmen, Faculty of Sciences and Arts, Amasya University, Turkey.
(3) Olivier Heubo-Kwegna, Saginaw Valley State University, USA.
Complete Peer review History: http://sciencedomain.org/review-history/15232

Received: ***29th April 2016***
Accepted: ***14th June 2016***
Published: ***30th June 2016***

Original Research Article

Abstract

In this work, cofinitely g-supplemented modules are defined and investigated some properties of these modules. It is shown that an arbitrary sum of cofinitely g-supplemented modules is cofinitely g-supplemented. In addition, amply cofinitely g-supplemented modules are also defined and given some equivalences.

Keywords: *G-small submodules; supplemented modules; g-supplemented modules; amply g-supplemented modules.*

2010 Mathematics Subject Classification: 16D60, 16D80.

1 Introduction

Throughout this paper all rings have an identity and all modules are unital left modules.

Let R be a ring and M be an R-module. We denote a submodule N of M by $N \leq M$. If M/N is finitely generated for $N \leq M$, then N is called a *cofinite submodule* of M. (See [1], [2], [3]) Let M be an R-module and $T \leq M$. If $K = 0$ for every $K \leq M$ with $T \cap K = 0$,

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then T is called an *essential submodule* of M and it is denoted by $T \trianglelefteq M$. Let $L \leq M$. If for every $T \leq M$ with $M = L + T$ implies that $T = M$, then L is called a *small submodule of* M and denoted by $L \ll M$. K is called a *generalized small* (briefly, *g-small*) *submodule* of M if for every $T \trianglelefteq M$ with $M = K + T$ implies that $T = M$, this is written by $K \ll_g M$ (in [4], it is called an *e-small submodule* of M and denoted by $K \ll_e M$). If T is both essential and maximal submodule of M, then T is called a *generalized maximal submodule* of M. The intersection of all generalized maximal submodules of M is called the *generalized radical* of M and it is denoted by $Rad_g M$ (in [4], it is denoted by $Rad_e M$). If M have no generalized maximal submodules, then the generalized radical of M is defined by $Rad_g M = M$. Let U and V be submodules of M. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. If $M = U + V$ and $M = U + T$ with $T \trianglelefteq V$ implies that $T = V$, or equivalently, $M = U + V$ and $U \cap V \ll_g V$, then V is called a *g-supplement* of U in M. If every submodule of M has a supplement in M, then M is called a *supplemented module*. (See [5], [6], [7], [8]) M is called a *g-supplemented module*, if every submodule of M has a g-supplement in M. (See [9], [10], [11], [12]) Let $U \leq M$. If for every $V \leq M$ such that $M = U + V$, U has a g-supplement V' with $V' \leq V$, we say U has *ample g-supplements* in M. If every submodule of M has ample g-supplements in M, then M is called an *amply g-supplemented module*.

There are some important properties of g-small submodules in [4], [9], [10] and [11].

Lemma 1.1. *Let M be an R-module and K, $N \leq M$. The following conditions are hold.* (*See* [4], [11])

(1) *If $K \leq N$ and $N \ll_g M$, then $K \ll_g M$.*
(2) *If $K \ll_g N$, then K is an g-small submodule in submodules of M which contain N.*
(3) *If $f : M \to N$ be an R-module homomorphism and $K \ll_g M$, then $f(K) \ll_g N$.*
(4) *If $K \ll_g L$ and $N \ll_g T$ for $L, T \leq M$, then $K + N \ll_g L + T$.*

Lemma 1.2. *Let M be an R-module. Then $Rad_g M = \sum_{L \ll_g M} L$.* (*See* [9])

Lemma 1.3. *Let M be an R-module, $X \leq U \leq M$ and V be a g-supplement of U. Then $(V + X)/X$ is a g-supplement of U/X in M/X.* (*See* [9])

2 Cofinitely G-supplemented Modules

Definition 2.1. Let M be an R-module. If every cofinite submodule of M has a g-supplement in M, then M is called a cofinitely g-supplemented module.

Clearly we see that every g-supplemented module is cofinitely g-supplemented, but the converse is not true in general.

Lemma 2.1. *Assume M be a finitely generated R-module. If M is cofinitely g-supplemented, then M is g-supplemented.*

Proof. Clear, since every submodule of M is cofinite. □

Lemma 2.2. *Let M be a cofinitely g-supplemented module. Then every factor module of M is cofinitely g-supplemented.*

Proof. Let M/X be any factor module of M and U/X be a cofinite submodule of M/X. Since $\frac{M}{U} \cong \frac{M/X}{U/X}$, U is a cofinite submodule of M. Since M is cofinitely g-supplemented, U has a g-supplement V in M. Then by Lemma 1.3, $(V + X)/X$ is a g-supplement of U/X in M/X. Hence M/X is cofinitely g-supplemented. □

Corollary 2.3. *Any homomorphic image of a cofinitely g-supplemented module is cofinitely g-supplemented.*

Proof. Clear from Lemma 2.2. □

Proposition 2.1. *Let M be a cofinitely g-supplemented module. Then every cofinite submodule of M/Rad_gM is a direct summand.*

Proof. Let U/Rad_gM be a cofinite submodule of M/Rad_gM. Then U is a cofinite submodule of M. Since M is cofinitely g-supplemented, U has a g-supplement V in M. Hence $M = U + V$ and $U \cap V \ll_g V$. Then $\frac{M}{Rad_gM} = \frac{U}{Rad_gM} + \frac{V+Rad_gM}{Rad_gM}$. Since $U \cap V \ll_g V$, by Lemma 1.1 and Lemma 1.2, $U \cap V \leq Rad_gM$. Hence $\frac{U}{Rad_gM} \cap \frac{V+Rad_gM}{Rad_gM} = \frac{U \cap V+Rad_gM}{Rad_gM} = 0$ and U/Rad_gM is a direct summand of M/Rad_gM. □

Lemma 2.4. *Let M be an R-module, $M_1 \leq M$, U be a cofinite submodule of M and M_1 be a cofinitely g-supplemented module. If $M_1 + U$ has a g-supplement in M, then so does U.*

Proof. Let X be a g-supplement of $M_1 + U$ in M. Then $M_1 + U + X = M$ and $(M_1 + U) \cap X \ll_g X$. Since U is a cofinite submodule of M, $U + X$ is also a cofinite submodule of M. Then by $\frac{M_1}{M_1 \cap (U+X)} \cong \frac{M_1+U+X}{U+X} = \frac{M}{U+X}$, $M_1 \cap (U + X)$ is a cofinite submodule of M_1. Since M_1 is cofinitely g-supplemented, $M_1 \cap (U + X)$ has a g-supplement Y in M_1, i.e. $M_1 \cap (U + X) + Y = M_1$ and $M_1 \cap (U + X) \cap Y \ll_g Y$. Following this, we have $M = M_1 + U + X = M_1 \cap (U + X) + Y + U + X = U + X + Y$ and $U \cap (X + Y) \leq X \cap (U + Y) + Y \cap (U + X) \leq X \cap (M_1 + U) + Y \cap M_1 \cap (U + X) \ll_g X + Y$. Hence $X + Y$ is a g-supplement of U in M. □

Corollary 2.5. *Let M be an R-module, U be a cofinite submodule of M and $M_i \leq M$ for $i = 1, 2, ..., n$. If $U + M_1 + M_2 + ... + M_n$ has a g-supplement in M and M_i is a cofinitely g-supplemented module for every $i = 1, 2, ..., n$, then U has a g-supplement in M.*

Proof. Clear from Lemma 2.4. □

Lemma 2.6. *Any sum of cofinitely g-supplemented modules is cofinitely g-supplemented.*

Proof. Let $\{M_i\}_{i \in I}$ be family of cofinitely g-supplemented submodules of an R-module M and $M = \sum_{i \in I} M_i$. Let U be any cofinite submodule of M. Since U is cofinite submodule of M, there exists a finite subset $\{i_1, i_2, ..., i_n\}$ of I such that $M = U + M_{i_1} + M_{i_2} + ... + M_{i_n}$. Since $U + M_{i_1} + M_{i_2} + ... + M_{i_n}$ has a g-supplement 0 in M and M_{i_k} is cofinitely g-supplemented for $k = 1, 2, ..., n$, then by Corollary 2.5, U has a g-supplement in M. □

Example 2.7. *Consider the $\mathbb{Z}$-module ${}_{\mathbb{Z}}\mathbb{Q}$. Since $Rad_{\mathbb{Z}}\mathbb{Q} = {}_{\mathbb{Z}}\mathbb{Q}$, ${}_{\mathbb{Z}}\mathbb{Q}$ have no proper cofinite submodules. Hence ${}_{\mathbb{Z}}\mathbb{Q}$ is a cofinitely g-supplemented. But it is well known that ${}_{\mathbb{Z}}\mathbb{Q}$ is not g-supplemented.*

3 Amply Cofinitely G-supplemented Modules

Definition 3.1. Let M be an R-module. If every cofinite submodule of M has ample g-supplements in M, then M is called an amply cofinitely g-supplemented module.

Lemma 3.1. *Let M be an R-module. If every submodule of M is cofinitely g-supplemented, then M is amply cofinitely g-supplemented.*

Proof. Let $M = U + V$ with $V \leq M$ and U is a cofinite submodule of M. Since $\frac{V}{U \cap V} \cong \frac{U+V}{U} = \frac{M}{U}$, $U \cap V$ is a cofinite submodule of V. Since V is cofinitely g-supplemented, $U \cap V$ has a supplement T in V. Because of this $V = U \cap V + T$ and $U \cap V \cap T \ll_g T$. Thus $M = U + V = U + U \cap V + T = U + T$ and $U \cap T = U \cap V \cap T \ll_g T$. Hence U has ample g-supplements in M and M is amply cofinitely g-supplemented. □

Corollary 3.2. *Every R-module is cofinitely g-supplemented if and only if every R-module is amply cofinitely g-supplemented.*

Proof. Clear from Lemma 3.1. □

Lemma 3.3. *If M is a π-projective and cofinitely g-supplemented module, then M is an amply cofinitely g-supplemented module.*

Proof: Let $M = U + V$, U be a cofinite submodule of M and X be a g-supplement of U. Since M is π -projective and $M = U + V$, there exists an R-module homomorphism $f : M \to M$ such that $Imf \subset V$ and $Im(1 - f) \subset U$. So, we have $M = f(M) + (1 - f)(M) = f(U) + f(X) + U = U + f(X)$. Suppose that $a \in U \cap f(X)$. Since $a \in f(X)$, then there exists $x \in X$ such that $a = f(x)$. Since $a = f(x) = f(x) - x + x = x - (1 - f)(x)$ and $(1 - f)(x) \in U$ we have $x = a + (1 - f)(x)$ and $x \in U$. Thus $x \in U \cap X$ and so $f(x) \in f(U \cap X)$. Therefore we have $U \cap f(X) \leq f(U \cap X) <<_g f(X)$. This means that $f(X)$ is a g-supplement of U in M with $f(X) \subset V$. Therefore M is amply g-supplemented.

Corollary 3.4. *If M is a projective and cofinitely g-supplemented module, then M is an amply cofinitely g-supplemented module.*

Proof. Clear from Lemma 3.3. □

Lemma 3.5. *Let M be an amply cofinitely g-supplemented module. Then every factor module of M is amply cofinitely g-supplemented.*

Proof. Let M/T be any factor module of M and let U/T be a cofinite submodule of M/T. Assume $\frac{M}{T} = \frac{U}{T} + \frac{V}{T}$ with $T, V \leq M$. Since U/T is a cofinite submodule of M/T, U is a cofinite submodule of M. Since $\frac{M}{T} = \frac{U}{T} + \frac{V}{T}$, $M = U + V$. Since M is amply cofinitely g-supplemented, U has a g-supplement K in M with $K \leq V$. Then by Lemma 1.3, $(K + T)/T$ is a g-supplement of U/T in M/T with $(K + T)/T \leq V/T$. Hence M/T is amply cofinitely g-supplemented. □

Corollary 3.6. *Let M be an amply cofinitely g-supplemented module. Then every homomorphic image of M is amply cofinitely g-supplemented.*

Proof. Clear from Lemma 3.5. □

Proposition 3.1. *Let R be a ring. The following statements are equivalent.*

(a) *${}_RR$ is g-supplemented.*
(b) *${}_RR$ is amply g-supplemented.*
(c) *${}_RR$ is cofinitely g-supplemented.*
(d) *${}_RR$ is amply cofinitely g-supplemented.*
(e) *Every finitely generated R-module is g-supplemented.*
(f) *Every finitely generated R-module is cofinitely g-supplemented.*
(g) *$R^{(I)}$ is cofinitely g-supplemented for every index set I.*
(h) *$R^{(I)}$ is amply cofinitely g-supplemented for every index set I.*
(i) *Every R-module is cofinitely g-supplemented.*
(k) *Every R-module is amply cofinitely g-supplemented.*

Proof. $(a) \Longleftrightarrow (b)$ Clear from [9] Corollary 8, since ${}_RR$ is projective.

$(a) \Longleftrightarrow (c)$ Clear from Lemma 2.1.

$(c) \Longleftrightarrow (d)$ Clear from Corollary 3.4, since ${}_RR$ is projective.

$(a) \Longrightarrow (e)$ Assume M be a finitely generated R-module and let $M = \langle m_1, m_2, ..., m_n \rangle$. Then $M = Rm_1 + Rm_2 + ... + Rm_n$. Since ${}_RR$ is g-supplemented and Rm_i $(i = 1, 2, ..., n)$ is an homomorphic image of ${}_RR$, by [9] Corollary 4, Rm_i is g-supplemented. Then by [9] Corollary 3, M is g-supplemented.

$(e) \Longleftrightarrow (f)$ Obtained from Lemma 2.1.

$(f) \Longrightarrow (g)$ By hypothesis, ${}_RR$ is cofinitely g-supplemented. Because of this, by Lemma 2.6, $R^{(I)}$ is cofinitely g-supplemented for every index set I.

$(g) \Longleftrightarrow (h)$ Clear from Corollary 3.4, since $R^{(I)}$ is projective for every index set I.

$(g) \Longrightarrow (i)$ Clear from Corollary 2.3, since every R-module is ${}_RR$-generated.

$(i) \Longleftrightarrow (k)$ Obtained from Corollary 3.2.

$(i) \Longrightarrow (c)$ Clear. □

4 Conclusion

In this paper, first time, the notion of cofinitely g-supplemented module is introduced. With this notion we had obtained equalities that was shown in Proposition 3.1.

Competing Interests

Author has declared that no competing interests exist.

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Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
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