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On Dual Similar Partner Curves in Dual 3-space

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

In this paper, For dual numbers in dual 3-space ID^3, we define the new families of curves which are called dual similar curves in dual 3-space with variable transformation. Then, we prove some theorems and characterizations about this family, We show that a family of dual similar curves with wanishing curvatures in dual 3-space ID^3.

Keywords: Dual quaternionic frame; similar curve; dual space; vector differential equation.

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1 Introduction

In the differential geometry, special curves have an important role. Especially the partner curves, i.e., the curves which are related to each other at the corresponding points, have attracted the

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attention of many mathematicians. Well-known partner curves are Bertrand curves, which are defined by the property that at the corresponding points of two space curves the principal normal vectors are common. Bertrand partner curves are studied [1, 2, 3, 4]. Ravani and Ku transported the notion of Bertrand curves to the ruled surfaces and called them Bertrand offsets. Recently, a new type of special curves in E^3 have introduced similar curves and between the space curves ξ and η such that, at the correponding points of the curves. The tangent lines of ξ is the same of η , then ξ is a called a similar curve, and η similar partner curve of ξ, [5, 6]. The set of dual numbers $ID = \{\widehat{w} \mid \widehat{w} = w + \varepsilon w^*;\ w,\ w^* \in IR\ \}$ is a commutative ring a unit. W.K.Clifford introduced dual numbers (1849-1870). With the help of dual numbers, Yaglom descirebed geometrical objects in 3-dimensional space, [7].

In this study, we gdefine a dual similar curves in ID^3 by means of this dual curve then we give some characterizations of these curves. Using vanishing curvatures forms, we obtain a family of dual similar curves in ID^3.

2 Preliminaries

A dual number is a number of the form $\widehat{w} = w + \varepsilon w^*$, where $w,\ w^* \in IR$, and $\varepsilon = (0, 1)$ arbitrary "dual unit" satisfy, the relation $\varepsilon^2 = 0$, [8, 9]. Addition and scalar multiplication are the operation on ID defined by the real number. The system ID is a vector space with respect to addition and scalar multiplication. The product $\widehat{w} \otimes \widehat{u}$ is the element in ID obtained by multiplying $w + \varepsilon w^*$ and $u + \varepsilon u^*$ as if they were polynomials, then using the relation $\varepsilon^2 = 0$ to simplify the result.

$$\widehat{w} \otimes \widehat{u} = \widehat{w}\ \widehat{u} = (w + \varepsilon w^*) . (u + \varepsilon u^*) = wu + \varepsilon (wu^* + w^* u) .$$

Then the dual number $w + \varepsilon w^*$ divided by the dual number $u + \varepsilon u^*$ provided $u \neq 0$ can be defined as

$$\frac{\widehat{w}}{\widehat{u}} = \frac{w + \varepsilon w^*}{u + \varepsilon u^*} = \frac{w}{u} + \varepsilon \frac{w^* u - w u^*}{u^2}.$$

The set of

$$\begin{aligned} ID^3 &= ID \times ID \times ID \\ &= \{\widehat{w} \mid \widehat{w} = (w_1 + \varepsilon w_1^*) + (w_2 + \varepsilon w_2^*) + (w_3 + \varepsilon w_3^*)\} \\ &= \{\widehat{w} \mid \widehat{w} = (w_1, w_2, w_3) + \varepsilon\ (w_1^*, w_2^*, w_3^*)\} \\ &= \{\widehat{w} \mid \widehat{w} = w + \varepsilon w^*\} \end{aligned}$$

is a norm on the ring ID. For any $\widehat{w} = w + \varepsilon w^*,\ \widehat{u} = u + \varepsilon u^* \in ID^3$,the scalar or inner produt and the vector prodıct of $\widehat{w}$ and $\widehat{u}$ are defined by,respectively,

$$\langle\ \widehat{w},\ \widehat{u}\rangle = \langle\ w, u\ \rangle + \varepsilon (\langle\ w, u^* \rangle + \langle\ w^*,\ u\ \rangle) ,$$

$$\widehat{w} \wedge\ \widehat{u} = (\widehat{w}_2\ \widehat{u}_3 - \widehat{w}_3 \widehat{u}_2, \widehat{w}_3 \widehat{u}_1 - \widehat{w}_1 \widehat{u}_3, \widehat{w}_1 \widehat{u}_2 - \widehat{w}_2 \widehat{u}_1),$$

where $\widehat{w}_i = w_i + \varepsilon w_i^*,\ \widehat{u} = u_i + \varepsilon u_i^* \in ID,\ 1 \leq i \leq 3$. If $\widehat{w} \neq 0$, the norm $\|\widehat{w}\|$ of $\widehat{w} = w + \varepsilon w^*$ is defined by

$$\|\widehat{w}\| = \sqrt{\langle\ \widehat{w},\ \widehat{w}\ \rangle} = \|w\| + \varepsilon \frac{\langle\ w, w^* \rangle}{\|w\|}.$$

If every $w_i(t)$ and $w_i^*(t)$, $1 \leq i \leq 3$ real valued functions, are differentiable the dual space curve

$$\begin{aligned} \widehat{w} \quad &: \quad I \subset IR \to ID^3 \\ t \quad &\to \quad \widehat{w}\,(t) = w_1\,(t) + \varepsilon w_1^*\,(t) + w_2\,(t) + \varepsilon w_2^*\,(t) + w_3\,(t) + \varepsilon w_3^*\,(t) \\ &= \quad w\,(t) + \varepsilon w^*\,(t) \end{aligned}$$

in ID^3 is differentiable. We call the real part $w(t)$ the indicatrix of $\widehat{w}(t)$. The dual arc-length of the curve $\widehat{w}(t)$ from t_1 to t is defined as

$$\widehat{s} = \int_{t_1}^{t} \left\| \widehat{w}(t)' \right\| dt = \int_{t_1}^{t} \left\| w(t)' \right\| dt + \varepsilon \int_{t_1}^{t} \langle\, t, w^* \rangle = s + \varepsilon s^*, \tag{2.1}$$

where t is a unit tangent vector of $\widehat{w}(t)$. Now we will obtain equations relatively to the derivatives of dual Frenet vectors throughout in ID^3. Let

$$\begin{array}{ccc} \widehat{w} & : & I \subset IR \to ID^3 \\ s & \to & \widehat{w}(s) = w(s) + \varepsilon w^*(s) \end{array}$$

be a dual curve with the arc-length parameter s of the indicatrix. Then, the dual unit tangent vector of

$$\widehat{w}(s)\,\frac{d\widehat{w}}{d\widehat{s}} = \frac{d\widehat{w}}{ds}\frac{ds}{d\widehat{s}} = \widehat{t}$$

With the equation of (2.1), we obtain

$$\widehat{s} = s + \varepsilon \int_{t_1}^{t} \langle\, t, w^* \rangle dt.$$

And from this $\frac{d\widehat{s}}{ds} = 1 + \varepsilon\Delta$, where the prime denotes differentiation with respect to the arc-length s of indicatrix and $\Delta = \langle\, t, w^* \rangle$. Since $\widehat{t}$ has constant length 1, its differentiation with respect to $\widehat{s}$ is given vector $\frac{d\widehat{t}}{d\widehat{s}}$ is called curvature function of $\widehat{w}(s)$. We impose the restriction that the function $\widehat{\kappa} : I \longrightarrow ID$ is never pure dual. Then , the dual unit vector $\widehat{n} = \frac{1}{\widehat{\kappa}}\frac{d\widehat{t}}{d\widehat{s}}$ is called the principal normal of $\widehat{w}(s)$. The dual vector $\widehat{b}$ is called the binormal of $\widehat{w}(s)$. The dual vectors $\widehat{t},\ \widehat{n},\ \widehat{b}$ are called the dual Frenet frame of $\widehat{w}(s)$. The equalities relative to derivative of dual Frenet vectors $\widehat{t},\ \widehat{n},\ \widehat{b}$ throughout the dual space curve are written in the matrix form

$$\frac{d}{d\widehat{s}}\begin{bmatrix} \widehat{t} \\ \widehat{n} \\ \widehat{b} \end{bmatrix} = \begin{bmatrix} 0 & \widehat{\kappa} & 0 \\ -\widehat{\kappa} & 0 & \widehat{\tau} \\ 0 & -\widehat{\tau} & 0 \end{bmatrix}\begin{bmatrix} \widehat{t} \\ \widehat{n} \\ \widehat{b} \end{bmatrix} \tag{2.2}$$

where $\widehat{\kappa} = \kappa + \varepsilon\kappa^*$ is pure dual curvature and $\widehat{\tau} = \tau + \varepsilon\tau^*$ is pure dual torsion. (2.2) are called the Frenet formulae of dual curve in ID^3, [8].

3 Dual Similar Curves in ID^3 with Variable Transformations

Definition 3.1. Let $\widehat{\alpha}(\ \widehat{s})$ be the dual curves in ID^3 parameterized by its arc-length $\widehat{s}$ and $\widehat{\alpha_*}(\widehat{s}_*)$ the dual similar partner curve of $\widehat{\alpha}(\ \widehat{s})$ with an arc-length parameter $\widehat{s}_*$, with non-zero curvatures $\widehat{\kappa}(\widehat{s}), \widehat{\tau}(\widehat{s}), \widehat{\kappa}_*(\widehat{s}_*), \widehat{\tau}_*(\widehat{s}_*)$ and the Frenet frame fied $\left\{\widehat{t}(\widehat{s}),\ \widehat{n}(\widehat{s}),\ \widehat{b}(\widehat{s})\right\}$ and $\left\{\widehat{t_*}(\widehat{s}_*),\ \widehat{n_*}(\widehat{s}_*),\ \widehat{b_*}(\widehat{s}_*)\right\}$, respectively. If there exists a variable transformation

$$\widehat{s} = \int \lambda_{\widehat{\alpha_*}}^{\widehat{\alpha}}(\ \widehat{s})d\ \widehat{s}_*$$

and corresponding relationship between the dual space curves $\widehat{\alpha}$ and $\widehat{\alpha_*}$ such that, at the corresponding points of the dual curves, the tangent vector of $\widehat{\alpha}(\ \widehat{s})$ is equal to the tangent vector of $\widehat{\alpha_*}(\widehat{s}_*)$, In this case the tangent vectors are the same for two curves i.e.,

$$\widehat{t}(\widehat{s}) = \widehat{t_*}(\widehat{s}_*) \tag{3.1}$$

for all corresponding values of parameters under the transformation $\lambda_{\widehat{\alpha_*}}^{\widehat{\alpha}}$. Where $\lambda_{\widehat{\alpha_*}}^{\widehat{\alpha}}$ is arbitrary function of the arc-length. If we integrate the equality (3.1) we have the following important theorem:

Theorem 3.1. *The position vectors of the family dual similar curves in ID^3 with variable transformation can be written in the following form.*

$$\int \widehat{t}(\widehat{s}(\widehat{s}_*))d(\widehat{s}_*) = \int \widehat{t}_*(\widehat{s})\lambda_{\widehat{\alpha}}^{\widehat{\alpha_*}} d(\widehat{s}_*).$$

Theorem 3.2. *Let $\widehat{\alpha}(\ \widehat{s})$ be a dual curve parameterized by arc-length $\widehat{s}$. Provided that $\widehat{\alpha}(\widehat{\varphi}\ )$ be another parametrization of the curve with parameter $\widehat{\varphi} = \int \widehat{\kappa}(\widehat{s})d(\widehat{s})$. Then the unit tangent vector $\widehat{t}$ of $\widehat{\alpha}(\ \widehat{s})$ satisfies a dual vector differential equation of third order as follows:*

$$\left[\frac{1}{\gamma}\left[\widehat{t}\right]''\right]' + \left[\frac{1+\gamma^2}{\gamma}\right](\widehat{t})' + \frac{\gamma'}{\gamma}\widehat{t} = 0. \tag{3.2}$$

where

$$\gamma(\widehat{\varphi}) = \frac{\widehat{\tau}(\widehat{\varphi})}{\widehat{\kappa}(\widehat{\varphi})},\ (\widehat{t})' = \frac{d\widehat{t}}{d\widehat{\varphi}},\ \ (\ \widehat{t})'' = \frac{d^2\widehat{t}}{d\widehat{\varphi}^2}.$$

Proof. If we write derivatives given in (2.2) according to $\widehat{\varphi}$,

$$\begin{aligned}
\frac{d\widehat{t}}{d\widehat{\varphi}} &= \widehat{\kappa}\frac{1}{\widehat{\kappa}}\ \widehat{n} = \widehat{n} \\
\frac{d\widehat{n}}{d\widehat{\varphi}} &= -\widehat{\kappa}\frac{1}{\widehat{\kappa}}\widehat{t} + \widehat{\tau}\frac{1}{\widehat{\kappa}}\ \widehat{b} = -\widehat{t} + \gamma\ \widehat{b} \\
\frac{d\widehat{b}}{d\widehat{\varphi}} &= -\widehat{\tau}\frac{1}{\widehat{\kappa}}\ \widehat{b} = -\gamma\widehat{n}
\end{aligned} \tag{3.3}$$

respectively, where $\gamma(\widehat{\varphi}) = \frac{\widehat{\tau}(\widehat{\varphi})}{\widehat{\kappa}(\widehat{\varphi})}$. Then, corresponding matrix form of (3.3) can be obtained

$$\begin{bmatrix} \widehat{t}' \\ \widehat{n}' \\ \widehat{b}' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \gamma \\ 0 & -\gamma & 0 \end{bmatrix} \begin{bmatrix} \widehat{t} \\ \widehat{n} \\ \widehat{b} \end{bmatrix}. \tag{3.4}$$

From the first and second equation of new Frenet derivatives (3.4) we obtain

$$\widehat{b} = \frac{1}{\gamma(\widehat{\varphi})}\left(\widehat{t}'' + \widehat{t}\right). \tag{3.5}$$

Substituting the above equation in the last equation (3.4), we obtain a dual vector differential equation of third order (3.2) as desired. Where the representation of the position vector of an arbitrary space can be determined as follows

$$\widehat{\alpha}(\ \widehat{s}) = \int \frac{1}{\widehat{\kappa}(\widehat{\varphi})}\widehat{t}(\widehat{\varphi})d\ \widehat{\varphi}\ + C \ \text{ and } \widehat{\varphi} = \int \widehat{\kappa}(\widehat{\varphi})$$

is a constant vector. □

Theorem 3.3. *Let $\widehat{\alpha}(\ \widehat{s})$ and $\widehat{\alpha_*}(\ \widehat{s}_*)$ be dual curves in ID^3. Then $\widehat{\alpha}(\ \widehat{s})$ and $\widehat{\alpha_*}(\ \widehat{s}_*)$ are dual similar curves in ID^3 with variable transformation if and only if the principal normal vectors are the same for all curves*

$$\widehat{n}(\ \widehat{s}) = \widehat{n_*}(\ \widehat{s}_*) \tag{3.6}$$

under the particular variable transformation

$$\lambda_{\widehat{\alpha}}^{\widehat{\alpha_*}} = \frac{d\ \widehat{s}}{d\widehat{s}_*} = \frac{\widehat{\kappa}\ (\widehat{s})}{\widehat{\kappa}_*(\widehat{s}_*)} \tag{3.7}$$

of arc-lenths.

Proof. Let $\widehat{\alpha}(\widehat{s})$ and $\widehat{\alpha_*}(\widehat{s}_*)$ be dual similar curves in ID^3 with variable transformation . Then differentiating the equation (3.6) with respect to $(\widehat{s}_*)$ we obtain

$$\frac{d\widehat{s}}{d\widehat{s_*}}\widehat{\kappa}(\widehat{s})\widehat{n}(\widehat{s}) = \widehat{\kappa}_*(\widehat{s}_*)\widehat{n_*}(\widehat{s}_*).$$

□

From the above equation leads to the two equation (3.6) and (3.7). Conversly, Let $\widehat{\alpha}(\widehat{s})$ and $\widehat{\alpha_*}(\widehat{s}_*)$ be dual similar curves in ID^3 with variable transformation satisfaying (3.6) and (3.7).. By multipling (3.6) with $\widehat{\kappa}_*(\widehat{s}_*)$ and integrate the result equality with respect to $\widehat{s}_*$, we obtain

$$\int \widehat{\kappa}_*(\widehat{s}_*)\widehat{n_*}(\widehat{s}_*)d\widehat{s}_* = \int \widehat{\kappa}_*(\widehat{s}_*)\widehat{n_*}(\widehat{s}_*)\frac{d\widehat{s}_*}{d\widehat{s}}d\widehat{s}. \tag{3.8}$$

From the equations (3.6), (3.7) and (3.8) take the form

$$\int \widehat{\kappa}_*(\widehat{s}_*)\widehat{n_*}(\widehat{s}_*)d\widehat{s}_* = \int \widehat{\kappa}(\widehat{s})\widehat{n}(\widehat{s})d\widehat{s}$$

which means that $\widehat{\alpha}(\widehat{s})$ and $\widehat{\alpha_*}(\widehat{s}_*)$ are dual similar curves in ID^3 with variable transformation and proof is completed.

Theorem 3.4. *Let $\widehat{\alpha}(\widehat{s})$ and $\widehat{\alpha_*}(\widehat{s}_*)$ be dual curves in ID^3. Then $\widehat{\alpha}(\widehat{s})$ and $\widehat{\alpha_*}(\widehat{s}_*)$ dual similar curves in ID^3 with variable transformation if and only if the binormal vectors are the same for all curves*

$$\widehat{b}(\widehat{s}) = \widehat{b_*}(\widehat{s}_*) \tag{3.9}$$

under arbitrary variable transformation $\widehat{s}_* = \widehat{s}_*(\widehat{s})$ of the arc-lengths.

Proof. $\widehat{\alpha}(\widehat{s})$ and $\widehat{\alpha_*}(\widehat{s}_*)$ are dual similar curves in ID^3 with variable transformation. Then there are a variable transformation of arc-length such that the dual tangent vectors and the dual principal normal vectors are the same. From equation (3.1) and (3.6) we obtain

$$\widehat{b_*}(\widehat{s}_*) = \widehat{t_*}(\widehat{s}_*) \times \widehat{n_*}(\widehat{s}_*) = \widehat{t}(\widehat{s}) \times \widehat{n}(\widehat{s}) = \widehat{b}(\widehat{s}).$$

□

Conversly, let $\widehat{\alpha}(\widehat{s})$ and $\widehat{\alpha_*}(\widehat{s}_*)$ are dual curvesin ID^3 which the same dual binormal vector under the arbitrary variable transformation $\widehat{s}_* = \widehat{s}_*(\widehat{s})$ of the arc-lengths. Differentiating the equation (3.9) with respect to $\widehat{s}_*$ we obtain

$$\left\{ \begin{array}{c} -\widehat{\tau}_*(\widehat{s}_*)\widehat{n_*}(\widehat{s}_*) = -\widehat{\tau}(\widehat{s})\widehat{n}(\widehat{s})\frac{d\widehat{s}}{d\widehat{s}_*} \\ \widehat{n_*}(\widehat{s}_*) = \widehat{n}(\widehat{s}) \end{array} \right\} \tag{3.10}$$

from equation (3.10) we obtain

$$\widehat{t_*}(\widehat{s}_*) = \widehat{n_*}(\widehat{s}_*) \times \widehat{b_*}(\widehat{s}_*) = \widehat{n}(\widehat{s}) \times \widehat{b}(\widehat{s}) = \widehat{t}(\widehat{s}).$$

Theorem 3.5. *Let $\widehat{\alpha}(\widehat{s})$ and $\widehat{\alpha_*}(\widehat{s}_*)$ are dual curves in ID^3. Then $\widehat{\alpha}(\widehat{s})$ and $\widehat{\alpha_*}(\widehat{s}_*)$ are dual similar curves with variable transformation if and only if ratios of torsion and curvature are the same for all curves*

$$\frac{\widehat{\kappa}_*(\widehat{s}_*)}{\widehat{\kappa}(\widehat{s})} = \frac{\widehat{\tau}_*(\widehat{s}_*)}{\widehat{\tau}(\widehat{s})}, \tag{3.11}$$

under the particular variable transformations $\left(\lambda_{\widehat{\alpha}}^{\widehat{\alpha_*}} = \frac{d\widehat{s}}{d\widehat{s}_*} = \frac{\widehat{\kappa}(\widehat{s})}{\widehat{\kappa}_*(\widehat{s}_*)}\right)$ keeping equal total curvatures, i.e.,

$$\widehat{\varphi}_*(\widehat{s}_*) = \int \widehat{\kappa}_*(\widehat{s}_*)d\widehat{s}_* = \int \widehat{\kappa}(\widehat{s})d\widehat{s} = \widehat{\varphi}(\widehat{s}) \text{ of the arc-lengths.} \tag{3.12}$$

Proof. Let $\widehat{\alpha}(\widehat{s})$ and $\widehat{\alpha_*}(\widehat{s}_*)$ are dual curves in ID^3. Then there exists a variable transformation of the arc-lengths such that the tangent and the binormal vectors are the same (definition 3.1. and theorem 3.4.). Differentiating the equations (3.1) and (3.9), we have

$$\widehat{\kappa}_*(\widehat{s}_*)\widehat{n_*}(\widehat{s}_*) = \widehat{\kappa}(\widehat{s})\widehat{n}(\widehat{s})\frac{d\widehat{s}}{d\widehat{s}_*} \tag{3.13}$$

$$-\widehat{\tau}_*(\widehat{s}_*)\widehat{n_*}(\widehat{s}_*) = -\widehat{\tau}(\widehat{s})\widehat{n}(\widehat{s})\frac{d\widehat{s}}{d\widehat{s}_*} \tag{3.14}$$

□

which leads to the following two equations

$$\widehat{\kappa}_*(\widehat{s}_*) = \widehat{\kappa}(\widehat{s})\frac{d\widehat{s}}{d\widehat{s}_*} \tag{3.15}$$

$$\widehat{\tau}_*(\widehat{s}_*) = \widehat{\tau}(\widehat{s})\frac{d\widehat{s}}{d\widehat{s}_*} \tag{3.16}$$

The variable transformation (3.11) is the equation (3.13) after integration. Dividing the above equations (3.13) and (3.16), we obtain the equation (3.11) under the variable transformations (3.12). Conversely, Let $\widehat{\alpha}(\widehat{s})$ and $\widehat{\alpha_*}(\widehat{s}_*)$ are dual curves in ID^3 such that the equation (3.11) is satisfied under the variable transformation (3.12) of the arclengths. Let now consider Theorem 3.2, the dual tangent vectors $\widehat{t_*}(\widehat{s}_*)$ and $\widehat{t}(\widehat{s})$ of the curves satisfy the following vector differential equations of third order as follows:

$$\left[\frac{1}{\gamma(\widehat{\varphi}(\widehat{s}))}\left[\widehat{t}(\widehat{\varphi}(\widehat{s}))\right]''\right]' + \left[\frac{1+(\gamma(\widehat{\varphi}(\widehat{s})))^2}{\gamma(\widehat{\varphi}(\widehat{s}))}\right](\widehat{t}(\widehat{\varphi}(\widehat{s})))' + \frac{(\gamma(\widehat{\varphi}(\widehat{s})))'}{\gamma(\widehat{\varphi}(\widehat{s}))}\widehat{t}(\widehat{\varphi}(\widehat{s})) = 0. \tag{3.17}$$

$$\left[\frac{1}{\gamma_*(\widehat{\varphi}(\widehat{s}_*))}\left[\widehat{t_*}(\widehat{\varphi}(\widehat{s}_*))\right]''\right]' + \left[\frac{1+(\gamma_*(\widehat{\varphi}(\widehat{s}_*)))^2}{\gamma_*(\widehat{\varphi}(\widehat{s}_*))}\right](\widehat{t_*}(\widehat{\varphi}(\widehat{s}_*)))' + \frac{(\gamma_*(\widehat{\varphi}(\widehat{s}_*)))'}{\gamma_*(\widehat{\varphi}(\widehat{s}_*))}\widehat{t_*}(\widehat{\varphi}(\widehat{s}_*)) = 0. \tag{3.18}$$

where

$$\gamma(\widehat{\varphi}(\widehat{s})) = \frac{\widehat{\tau}(\widehat{\varphi}(\widehat{s}))}{\widehat{\kappa}(\widehat{\varphi}(\widehat{s}))}, \quad \gamma_*(\widehat{\varphi}(\widehat{s})) = \frac{\widehat{\tau}(\widehat{\varphi}(\widehat{s}_*))}{\widehat{\kappa}(\widehat{\varphi}(\widehat{s}_*))}, \quad \widehat{\varphi} = \int \widehat{\kappa}(\widehat{s}) \text{ and } \widehat{\varphi}_* = \int \widehat{\kappa}(\widehat{s}_*) .$$

The equation (3.13) causes

$$\gamma_*(\widehat{\varphi}(\widehat{s})) = \gamma(\widehat{\varphi}(\widehat{s}))$$

under the variable transformations $\widehat{\varphi}(\widehat{s}) = \widehat{\varphi}(\widehat{s}_*)$. So that the equations (3.17) and (3.18) under the equation (3.11) and the transformation (3.12) are the same. Hence the solution is the same, i.e., the tangent vectors are the same which completes the proof of the theorem.

Example 3.6. *Euler Spirals were discovered indepently by three researchers. In 1694, Bernoulli wrote the equations for the Euler spiral for the first time, but did not draw the spirals or compute them numerically. In 1774, Euler re-described their properties, and derived a series expansion to the curve's integrals. Later, In 1871, he also computed the spiral's end points. The curves were re-discovered 1891 for the third time by Talbot, who used them to design railway tracks [10]. An Euler spiral is curve whose curvature changes linearly with its curve length (the curvature of a cicular curve is equal to the reciprocal of the radius). Euler spirals are also commonly referred to as spiros, clothoids or Cornu spirals. Moreover, Euler spiral in railroad/highway engineering for connecting and transiting the geometry between a tangent and a circular curve. Let us consider the Euler Spiral* $\overrightarrow{\gamma}(s) = (\gamma_1(s),\ \gamma_2(s),\ \gamma_3(s))$ *of* E^3

$$\left\{\begin{array}{c} \gamma_1(\widehat{s}) = \frac{3}{5}\int \sin(\widehat{s}^2+1)d\widehat{s} \\ \gamma_2(\widehat{s}) = \frac{3}{5}\int \cos(\widehat{s}^2+1)d\widehat{s} \\ \gamma_3(\widehat{s}) = \frac{4}{5}\widehat{s} \end{array}\right\}$$

Then the dual unit tangent vector $\widehat{t}$ of $\gamma(s)$ satisfies a vector differential equation of third order given by

$$\left[\frac{1}{\gamma(\widehat{\theta})}\left[\widehat{t}(\widehat{\theta})\right]''\right]' + \left[\frac{1+\gamma^2(\widehat{\theta})}{\gamma(\widehat{\theta})}\right](\widehat{t}(\widehat{\theta}))' + \frac{\gamma'(\widehat{\theta})}{\gamma(\widehat{\theta})}\widehat{t}(\widehat{\theta}) = 0.$$

Proof. We are calculated this curve's curvature function with help of Mathematica Programme $\widehat{\kappa} = \frac{6\widehat{s}}{5}$ and $\widehat{\tau} = -\frac{8\widehat{s}}{5}$. The Frenet -Serret frame of the curve $\gamma = \gamma(s)$ may be written by the aid Mathematica Programme as fallows: □

$$\left\{\begin{array}{l} \widehat{t}(\widehat{s}) = (\frac{3}{5}\sin(\widehat{s}^2+1),\ \frac{3}{5}\cos(\widehat{s}^2+1),\ \frac{4}{5}), \\ \widehat{n}(\widehat{s}) = (\ \cos(\widehat{s}^2+1),\ -\sin(\widehat{s}^2+1),\ 0), \\ \widehat{b}(\widehat{s}) = (\ \frac{4}{5}\sin(\widehat{s}^2+1), \cos(\widehat{s}^2+1), -\frac{3}{5}). \end{array}\right\}$$

Lemma 3.7. *The family of Euler Spiral Forms a family of dual similar curves with variable transformations. We can deduce the position vector of Euler Spiral curve using the definition of dual similar curves with variable transformation as follows:*

$$\begin{aligned} \gamma(\widehat{v}) &= (\gamma_1(\widehat{v}),\ \gamma_2(\widehat{v}),\ \gamma_3(\widehat{v})) \\ &= (\frac{3}{5}\int \sin(\widehat{v}^2+1)d\widehat{v},\ \frac{3}{5}\int \cos(\widehat{v}^2+1)d\widehat{v},\ \frac{4}{5}\widehat{v}) \end{aligned}$$

where $\widehat{s} = \widehat{v}$ *is the arc-length of Euler Spiral and the curvature is* $\widehat{\kappa}_x(\widehat{v}) = \frac{6\widehat{v}}{5}$. *The dual unit tangent vectors of Euler Spiral takes the form:*

$$\widehat{t}(\widehat{v}) = (\frac{3}{5}\sin(\widehat{v}^2+1),\ \frac{3}{5}\cos(\widehat{v}^2+1),\ \frac{4}{5}),$$

from theorem 3.1., we can write as the following,

$$\gamma_*(\widehat{s}_*) = \int \left[\frac{3}{5}\sin\left[(\widehat{v}(\widehat{s}_*))^2+1\right],\ \frac{3}{5}\cos\left[(\widehat{v}(\widehat{s}_*))^2+1\right],\ \frac{4}{5}\right] d\widehat{s}_*$$

where $\widehat{s}_* = \widehat{s}$. *From the equation (3.15), we obtain*

$$d\widehat{s} = \frac{\widehat{\kappa}_*}{\widehat{\kappa}} d\widehat{s}_*$$

or

$$\widehat{s}(\widehat{s}_*) = \int \frac{\widehat{\kappa}_*}{\widehat{\kappa}} d\widehat{s}_*.$$

If we put the curvature $\widehat{\kappa}_* = \widehat{\kappa}(\widehat{s})$, $\quad \widehat{s}_* = \widehat{s}$, *we have* $\widehat{v}(\widehat{s}_*) = \int \widehat{\kappa}(\widehat{s})d\widehat{s}$. *Then the position vector of Euler Spiral with arbitrary curvature* $\widehat{\kappa}(\widehat{s})$ *takes the following form:*

$$\gamma(\widehat{v}) = \int (\frac{3}{5}\sin(\int \widehat{\kappa}(\widehat{s})d\widehat{s}+1),\ \frac{3}{5}\cos(\int \widehat{\kappa}(\widehat{s})d\widehat{s}+1),\ \frac{4}{5})d\widehat{s}.$$

4 Conclusion

In dual 3-space ID, the dual similar curves are defined and some properties of these curves are obtained. It is shown that dual curves with vanishing curvatures form the families of dual similar curves.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Neill BO. Semi-riemannian geometry with applications to relativity. Academic Press Inc., London; 1983.

[2] Izumiya S, Takeuchi N. New special curves and developable surfaces. Turk. J.Math. 2004;28:153-163.

[3] Salkowski E. Zur transformation von Raumkurven. Mathematiche Annalen.1909;66(4)517-557.

[4] Ch. Bioche. Sur les courbes de M. Bertrand. Bull. Soc. Math. France. 1889;17:109-112.

[5] Bababadag F. On similar partner curves in bishop frames with variable transformations. International Journal Of Engineering Research and Science (IJNTR). 2015;1(6):59-64.

[6] Ali AT. Position vectors of general helices in Euclidean 3-space. Bull. Math. Anal. 2010;3(2):198-205.

[7] Yaglom E. Geometrie der dynamen. Leipzig, in German; 1903.

[8] Köse Ö, Nizamoglu Ş, Sezer M. An explicit characterization of dual spherical curves. Doga Mat. 1998;12(3):105-113.

[9] Veldkamp GR. On the use of dual numbers, vectors and matrices instantaneous, spetial kinematics. Mech. Mach. Thory. 1967;11(2):141-156.

[10] Monterde J. Salkowski curves revisted: A family of curve with constant curvature and non-constant torsion. Comput. Aided Geo. Design. 2009;26:271-278.

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