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Some Properties of Extended Ladder Links and Component Numbers

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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Abstract

We define a general class of planar graphs called extended ladders and their corresponding extended ladder links. We then give some of the properties of the extended ladder graphs and corresponding extended ladder links. Finally, we count the number of components of some extended ladder links.

Keywords: ladder graph; links; knots; component number; planar graph.

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1 Introduction

It is widely known in knot theory literature that a knot has a corresponding signed planar graph and that a signed planar graph also has a corresponding knot which depends on the signs of the edges of its signed planar graph. This implies that every knot universe has a corresponding unsigned planar graph. This provides a foundation of a solid relationship between knot theory and graph theory,

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and it allows for some of the notions in graph theory to be transferred to knot theory. It is also known that an isthmus with one vertex of degree one in a planar graph corresponds to a loop in the associated link, which is trivial or can be undone using Reidemeister move I. For example, any tree in graph theory corresponds to the unknot in knot theory. Since we are interested in knots or links in this paper, a graph will have no isthmuses with a vertex of degree one unless otherwise stated. We refer the reader to [1]. Knot theory is concerned with telling which knots or links are equivalent and which are not equivalent, see [2, 3]. Knot invariants provide us with a mechanism that allows us to tell knots or links apart, see [1]. A knot invariant of a knot or link K is a mathematical quantity $\sigma(K)$ that is alloted to K such that $\sigma(K_i) = \sigma(K_j)$ wherever K_i and K_j are equivalent knots or links, see [4]. The component number is one of the famous knot invariants that is used as a tool to distinguish one knot from the other and its applications are best explored in [5, 6, 7]. The component number of a knot or link is applied in biological sciences, to determine the action of an enzyme on DNA, in particular, in the recombination of DNA, see [1]. The path-width of a graph has been discussed in [8]. It has applications in linguistics and it has been used extensively as a tool to study natural language processing, see [9]. In this paper we start by defining a general class of planar graphs called extended ladders and their corresponding extended ladder links. We then move on to giving some of the properties of the extended ladder graphs and corresponding extended ladder links. Finally, we compute the component number of some of the extended ladder links.

2 Characterization of Extended Ladder Graphs and their Corresponding Links

In this section we define a class of graphs called extended ladder graphs and a class of links called extended ladder links. Since this is a broad class of graphs, we give some examples of graphs and links belonging to this class. We conclude this section by giving the path-width of an extended ladder graph and its corresponding link. For graph theory concepts we refer to [10].

An n-chain is a graph which is isomorphic to a path P_n thus a 3-chain is isomorphic to P_3. Two chains P and Q will be said to intersect if they have a vertex in common which is not a leaf in at least one chain. A set of i-chains is called free if the chains are connected only at the end vertices (leaves) or not connected at all. Let v_1 and v_2 be vertices of a graph G, joined by two or more n_k-chains, where at least one of the chains is of size greater than one, then we call the n_k-chains parallel. Note that the sizes of the chains are irrelevant in this definition, that is, we can have chains of different sizes being parallel.

Definition 2.1. A graph G is called a *ladder* if it has the following set of vertices $V(G) = \{1, 2, \cdots, 2n\}$ and the edge set $E(G) = \{\{1, 2\}, \{2, 3\}, \cdots, \{n-1, n\}\} \cup \{\{n+1, n+2\}, \{n+2, n+3\} \cdots \{2n-1, 2n\}\} \cup \{\{1, 2n\}, \{2, 2n-1\}, \cdots, \{n, n+1\}\}$. The edges are of two types. An edge $e \in \{\{1, 2n\}\{2, 2n-1\}, \cdots, \{n-1, n+2\}\{n, n+1\}\}$ is called a *rung*. An edge $e \in \{\{1, 2\}, \{2, 3\}, \cdots, \{n-1, n\}\} \cup \{\{n+1, n+2\}, \{n+2, n+3\}, \cdots, \{2n-1, 2n\}\}$ is called an *s-bar*. The union of the edges which are s-bar's is a graph with two disjoint paths. A ladder graph is denoted by $L_{2,n}$, see [11]. An *extended ladder* graph is a ladder graph in which, each rung has been replaced by a parallel class of s_i-chains. We shall denote an extended ladder graph by $L_{2_{s_1,s_2,\cdots,s_n},n}$ where each s_i is representing the parallel class replacing rung i. This class of graphs is planar, hence we can talk about its corresponding link universes.

To clarify the definition, we give two examples of extended ladder graphs in the diagrams of Fig. 1, where $G_1 = L_{2_{q_1,q_2,q_3,q_4},4}$ and $G_2 = L_{2_{q_1,q_2,q_3,q_4,q_5},5}$.

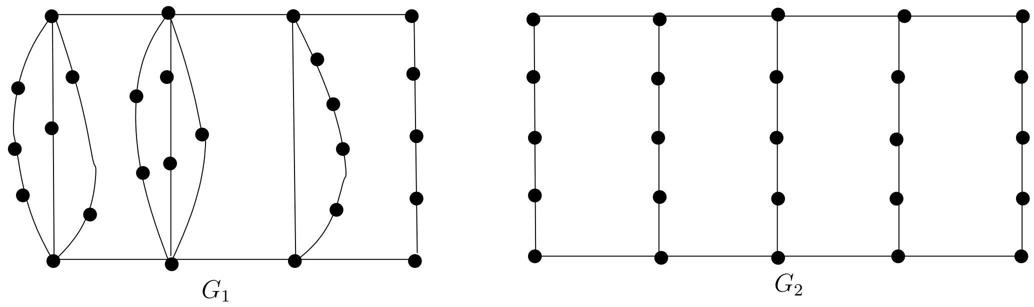


Fig. 1. Extended ladder graphs

Let G be a graph, we shall call a graph G' obtained by replacing every subgraph of G isomorphic to an i-chain, for some $i > 3$ by a 3-chain, a source graph of G. Consequently $K(G')$ is a source link for $K(G)$.

The following three definitions can be found in [12].

Definition 2.2. Let G be a graph and let $E_1(G)$ and $E_2(G)$ be edge-sets of G. A pair $(E_1(G), E_2(G))$ is an *n-separation* if $|V(E_1(G)) \cap V(E_2(G))| \leq n$. Let G be a graph with k edges, then G is said to be of *path-width n* if its edges can be ordered as $e_1, e_2, e_3, \cdots, e_k$ such that for all i, the pair $(\{e_1, e_2, e_3, \cdots, e_i\}, \{e_{i+1}, e_{i+2}, \cdots, e_k\})$ is an *n-separation.*

We shall denote the path-width of a graph G by $pw(G)$. The notion of *path-width* for planar graphs and their corresponding links has been shown that it is closely related to the notion of bridge number for links, see [12]. We shall denote a link corresponding to a planar graph G by $K(G)$.

Definition 2.3. A link diagram K has path-width n if its associated planar graph has path-width n.

It is clear that a link has different diagrams representing it. This points to the fact that a link corresponds to different planar graphs.

Definition 2.4. The *path-width of a link K*, denoted by $pw(K)$, is the least path-width of all diagrams of the link K.

The minimal-diagram path-width of a link K, denoted by $pw_m(K)$ is the least path-width of all minimal diagrams of the link K. The minimal-diagram path-width, $pw_m(K)$ is computable, since every K has a finite number of minimal diagrams and $pw(K) \leq pw_m(K)$. Moreover, $pw(K) = 1$ if and only if K is the unknot.

The following lemma clear and stated without proof

Lemma 2.1. Let G be a graph and G' its source graph. Then $pw(G) = pw(G')$.

Proposition 2.1. Let G be a graph with n-parallel free n_k-chains and let $G_{||}$ be the graph obtained from G by replacing the n-parallel free n_k-chains with 2-parallel free n_k-chains. Then the path-width of the graph G is equal to the path-width of the graph $G_{||}$.

Proof. Without loss of generality, consider the graphs G and $G_{||}$ shown in the diagram of Fig. 2.

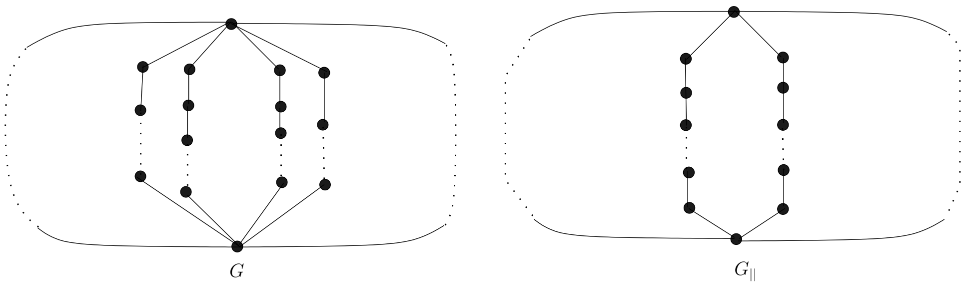


Fig. 2. A graph G and its reduced graph $G_{||}$

By applying Lemma 2.1 the path-widths of G and $G_{||}$ are equal to the path-widths of their source graphs, G' and $G'_{||}$, respectively, shown in Fig. 3.

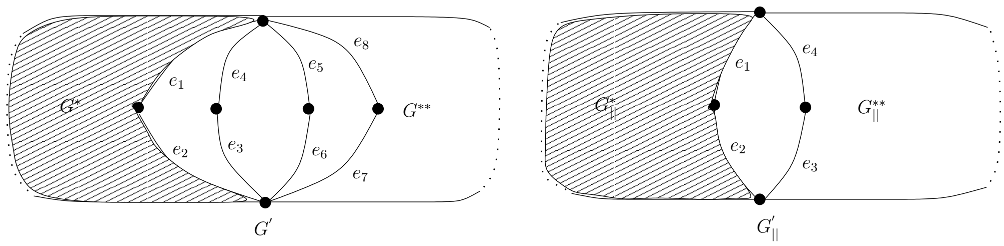


Fig. 3. A graph G and its source graph $G'_{||}$

Let G^* and $G^*_{||}$ be parts of G' and $G'_{||}$ which are identical, and let G^{**} and $G^{**}_{||}$ be parts of G' and $G'_{||}$ which are also identical. Consider the orderings $E(G^*), e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, E(G^{**})$ and $E(G^*_{||}), e_1, e_2, e_3, e_4, , E(G^{**}_{||})$ for the graphs G' and $G'_{||}$ respectively. In particular, we get

$$\begin{aligned} & |V(E(G^*) \cup \{e_1, \cdots, e_i\}) \cap V(E(G^{**}) \cup \{e_{i+1}, \cdots, e_8\})| \\ = \quad & |V(E(G^*_{||}) \cup \{e_1, \cdots, e_i\}) \cap V(E(G^{**}_{||}) \cup \{e_{i+1}, \cdots, e_4\})|. \end{aligned}$$

Hence the path-width of G' is equal to the path-width of $G'_{||}$ regardless of the number of k-chains. Consequently, the path-width of G is equal to the path-width of $G_{||}$. □

Theorem 2.1. Let G be an extended ladder graph. Then the path-width of G, $pw(G) = 3$.

Proof. Without loss of generality and applying Lemma 2.1 and Proposition 2.1 any extended ladder graph will reduce to a graph similar to graph G shown in Fig. 4.

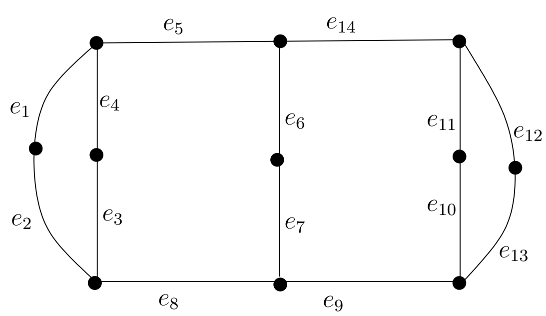


Fig. 4. Reduced source extended ladder graph

The path-width of the graph G is computed by following the order

$$e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}.$$

Therefore the path-width of G is equal to 3. □

Theorem 2.2. Let G be an extended ladder graph and $K(G)$ its corresponding extended ladder link. Then the path-width of $K(G)$, $pw(K(G)) \leq 3$.

Proof. By Theorem 2.1, the path-width of G is 3. Hence by Definition 2.3, the link diagram $K(G)$ has path-width 3. □

3 Component Number of Links Corresponding to Extended Ladder Graphs

In this section we will discuss the component numbers of some extended ladder graphs. According to Definition 2.1, this is a broad class of graphs. We are going to define a few classes of graphs contained in this broad class and compute the component number of links for these classes.

For a graph G, with the Tutte polynomial $T(G; x, y)$, it is well known that an evaluation $T(G; -1, -1) = (-1)^{E(G)}(-2)^{c-1}$, where c is the component number of the link whose universe is the medial graph of G. We can thus define, for a link diagram $K(G)$, a number $\Upsilon(K(G)) = (-1)^{E(G)}(-2)^{c-1}$, where c is the component number of $K(G)$ and $E(G)$ is the edge set of G. We state the following properties of the number $\Upsilon(K(G)) = (-1)^{E(G)}(-2)^{c-1}$ without proof, see [13].

Proposition 3.1. *Let G be a planar graph. Then*

(a) $\Upsilon(K(G)) = (-2)^{n-1}$, *if G is a graph with n vertices and no edges, thus* $\Upsilon(K(G)) = 1$ *if G is a single vertex,*
(b) $\Upsilon(K(G)) = -1$, *if G is an isthmus or a loop,*
(c) $\Upsilon(K(G)) = -\Upsilon(K(G\backslash e))$, *if e is a loop,*
(d) $\Upsilon(K(G)) = -\Upsilon(K(G/e))$, *if e is an isthmus, and*
(e) $\Upsilon(K(G)) = \Upsilon(K(G\backslash e)) + \Upsilon(K(G/e))$ *if e is neither a loop nor an isthmus.*

Corollary 3.1. Let G be a tree, then $K(G)$ has one component.

Theorem 3.1. If G is an unsigned planar graph then :

(a) $\Upsilon(K(G)) = \Upsilon(K(G/t/w))$ if t and w are a series pair in G.
(b) $\Upsilon(K(G)) = \Upsilon(K(G\backslash t\backslash w))$ if t and w are a parallel pair in G.

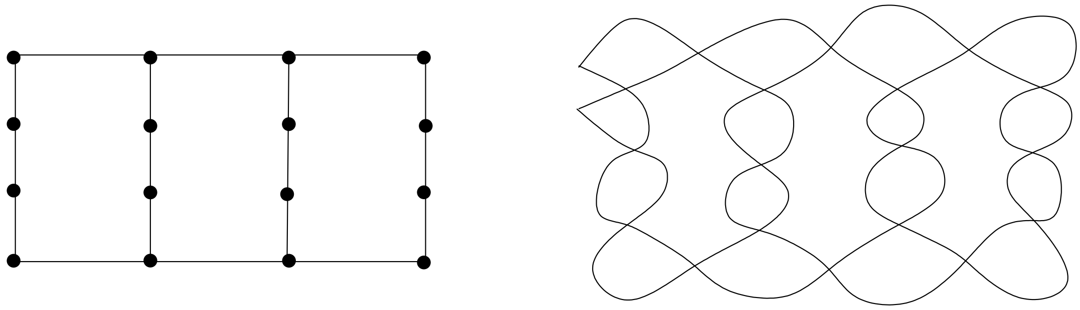
We also state the following theorem without proof, we refer to [11].

Theorem 3.2. Let G be a ladder graph, $L_{2,n}$, then the number of components of the link corresponding to G is

(a) 1 if n is odd,
(b) 2 if n is even.

Definition 3.1. An extended ladder graph $L_{2_{s_1,s_2,\cdots,s_n},n}$ where each s_i is a single chain of size q will be called a *uniform extended ladder graph.* We shall denote a uniform extended ladder graph by $L_{2_q,n}$.

An example of a uniform extended ladder graph and its link universe are shown in Fig. 5.



$L_{2_4,4}$ $K(L_{2_4,4})$

Fig. 5. Extended ladder graph and its corresponding link universe

We will consider various cases of component numbers of uniform extended ladder links $K(L_{2_q,n})$.

Proposition 3.2. Let G be the graph $L_{2_q,n}$ and let $K(G)$ be a link corresponding to the graph G. Then the component number of $K(G)$ is,

(a) n for any n and q even,

(b) 1 if n is odd and q is odd,

(c) 2 if n is even and q is odd.

Proof. Recall that q is the size of the extended rung, that is, the size of the chain.

(a) Let v_i and u_i be the two end vertices of the extended rung i of an extended ladder, $L_{2_q,n}$ and let q be even. Then we can form $\frac{q}{2}$ series pairs. By applying Theorem 3.1(a) we contract all series pairs, thus identifying vertex v_i and u_i for all $i \in \{1, 2, \cdots, n\}$. Hence the resulting minor is an n-path up to parallel class where all edges are parallel pairs. By applying Theorem 3.1(b) we delete all the parallel pairs resulting in n isolated vertices.

(b) Let n be odd and q be odd. Thus we can form a $\frac{q-1}{2}$ series pairs and there will be one edge remaining. By applying Theorem 3.1(a) we contract all series pairs. Hence the resulting minor is an n-ladder. By Theorem 3.2 an odd ladder has one component.

(c) Similar to proof of part (b) where we get an even ladder. By Theorem 3.2 we get two components. □

Notation 3.3. *We denote by $L_{2_{i_q},n}$ a uniform extended ladder graph with i parallel chains of size q.*

Theorem 3.4. Let G be the graph $L_{2_{i_q},n}$ and let $K(G)$ be a link corresponding to the graph G. Then the component number of $K(G)$ is

(a) $n \times i$ for q even and for all n and i,

(b) 2 for n even, q odd and any i,

(c) 2 for n odd, q odd and i even,

(d) 1 for n odd, q odd and i odd.

Proof.

(a) Let q be even, then we have an even number of edges for each chain. Without loss of generality, we demonstrate the procedure in the proof by diagrams of Fig. 6 and Fig. 7 where q is any even number, $n = 4$ and $i = 4$. We contract series pairs until each chain reduces to a 3-chain as shown in the diagram of Fig. 6.

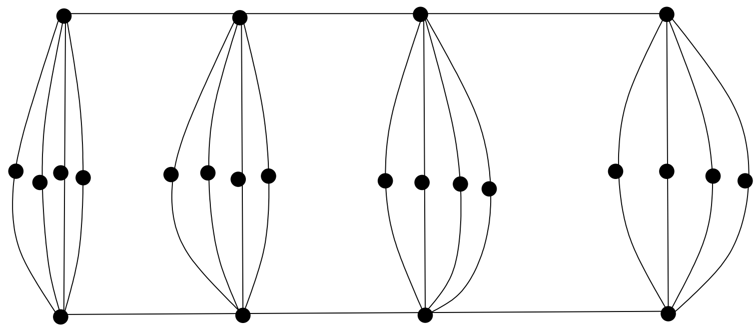


Fig. 6. Extended ladder graph with reduced parallel even chains

Hence if we contract one 3-chain on each bunch of reduced parallel chains, we get a minor shown in the diagram of Fig. 7.

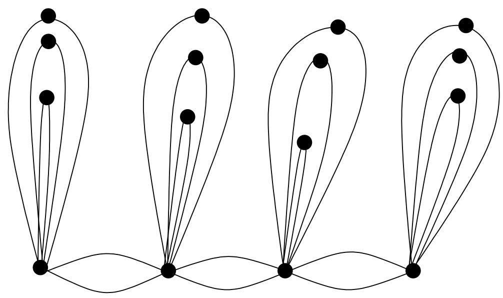


Fig. 7. Extended ladder graph with contracted parallel even chains

By applying Theorem 3.1(b) we delete all parallel pairs resulting in $n \times i$ isolated vertices. Hence the result.

(b) Let n be even, q be odd and i any positive integer. Since we have an odd number of edges, we contract series pairs until each chain reduces to a a single edge. Hence the resulting minor is an n-ladder with all rungs replaced by i parallel edges. By applying Theorem 3.1(b) we delete all parallel pairs resulting in two of the following two cases:

  (i) if i is odd, we get an n-ladder. Hence by applying Theorem 3.2(b) we get two components since n is even.

  (ii) if i is even, we get two disjoint paths. Hence by applying Corollary 3.1 we get one component for each path. Hence resulting in two components.

(c) Let n and q be odd. If i is odd, the proof is similar to part (b)(i) if i is even the proof is similar to part (b)(ii).

□

Notation 3.5. *We denote by $L^{j}_{2_{iq},n}$ a uniform extended ladder graph with i parallel chains of size q, where j bunches of the i parallel chains have been removed consecutively.*

Proposition 3.3. Let G be the graph $L^{j}_{2_{iq},n}$ and let $K(G)$ be a link corresponding to the graph G. Then

(a) if q is even the component number of $K(G)$ is $ni - ji$.

(b) if q is odd the component number of $K(G)$ is

 (i) 1 for n even, j and i odd,

 (ii) 1 for j even, n and i odd.

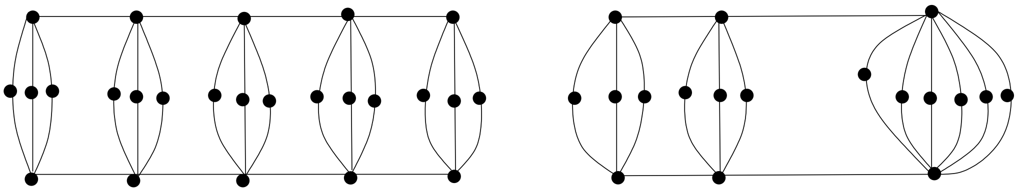
(c) if q is odd the component number of $K(G)$ is

 (i) 2 for j odd, n and i even,

 (ii) 2 for i odd, n and j even,

 (iii) 2 for n, j and i odd.

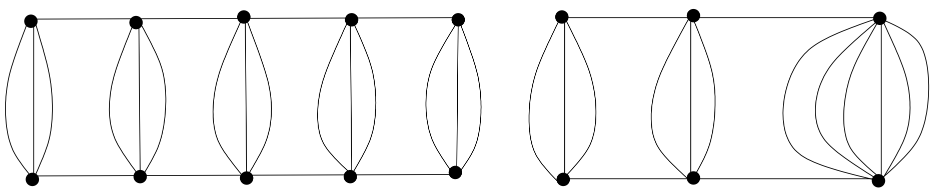
Proof. Let q be even, that is, we have an even number of edges for each chain. We contract series pairs until each chain reduces to a 3-chain similar to proof of Theorem 3.4(a). By removing j bunches of parallel edges, we have created $\frac{j}{2}$ series pairs on the two disjoint paths formed by the union of the edges of s-bar's of G. By Theorem 3.1(a), we contract all series pairs, without loss of generality, we obtain the minors shown in the diagrams of Fig. 8, depending on j, when $n = 5$, $i = 3$, $j = 2$ and $j = 3$.



j even. j odd.

Fig. 8. Extended ladder graph with reduced parallel even chains

Let q be odd, that is, we have an odd number of edges for each chain. We contract series pairs until each chain reduces to a single edge similar to proof of Theorem 3.4(b). By removing j bunches of parallel edges, we have created $\frac{j}{2}$ series pairs on the two disjoint paths formed by the union of the edges of s-bar's of G. By Theorem 3.1(a), we contract all series pairs, without loss of generality, we obtain the minors shown in the diagrams of Fig. 9, depending on j, when $n = 5$, $i = 3$, $j = 2$ and $j = 3$.



j even j odd

Fig. 9. Extended ladder graph with reduced parallel odd chains

The rest of the proof is similar to the proof of Theorem 3.4. ☐

4 Conclusion

The pathwidth and component numbers of extended ladder links are given in this paper. These links are closely related to the pretzel links hence for future research, extending some of the results for preztel links to these links is a possiblility.

Competing Interests

Authors have declared that no competing interests exist.

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